THE

TEACHING OF

ARITHMETIC

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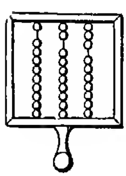

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Contents

Preface

Few educational experiences provide a thrill equal to that of "seeing through" a mathematical fact or process — that is, perceiving its simplicity and purpose. It has been the happy privilege of the writer to watch the growth of teachers' interest and enjoyment in arithmetic as they grasp the essential features of the subject. And it has been an even greater pleasure to see children thoroughly enjoy their arithmetic work as they acquire an understanding of the basic elements of mathematics.

This book presents some of the procedures which have made the teaching of the "teaching of arithmetic" to adults and the teaching of arithmetic to children a pleasant experience.

In these teaching procedures much emphasis is given to understanding. An understanding of arithmetic, the author believes, is in part dependent on a thorough knowledge of the number system and of the true role of numbers in making it easier for man to do his work. The teaching procedures described are further characterized by the author's assumption that if the mathematical generalizations of arithmetic are to be genuine possessions of the child, the child must have experiences that will make it possible for him to arrive at these generalizations. The work of the teacher then becomes more a case of creating situations in which the child can gain meaningful experiences, than of demonstrating or explaining mathematical facts and processes to the child.

In the preparation of these materials the author has become indebted to the many writers and research students whose

writings were used. Particular mention should be made of the children of the University Elementary School and the many graduate students at the State University of Iowa who have been in the writer's classes.

Special acknowledgment of indebtedness is made to members of the University Elementary School Staff as follows: to Miss Charlotte Junge for experimental work in developing the idea of tens and the number concepts program; to Miss Alice Hyslop for experimental work with the abacus and in mental arithmetic; to Miss Dorothy Gordon, Miss Ruth Willard, Miss Maxine Dunfee, and Miss Dorothy Welch for their work in putting into practice many of the procedures; and to Miss Esther Beamer for aid in the preparation of the manuscript.

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1

The Teaching of Arithmetic

THE IMPORTANCE OF UNDERSTANDING

The preface of an arithmetic textbook written fifty years ago stated that the teaching procedures in the book made for understanding of arithmetical facts and processes. Many earlier texts had referred to the importance of *understanding;* practically all arithmetic books written since that time have emphasized this aspect of the learning process. Thus, for years, "Teach with understanding" has been one of the maxims of arithmetical instruction. In spite of this emphasis, the understanding of those who study arithmetic has been unsatisfactory. In an effort to improve understanding, another maxim, "Teach with meaning," has recently been adopted. Yet today every experienced arithmetic teacher knows that children can correctly go through processes like multiplication and division without knowing what the processes mean.

What implications are students of the teaching of arithmetic to draw from the preceding statement? Are they to conclude that understanding is too difficult for most elementary school children to attain, or are they to conclude that the procedures now in use are inadequate? Before attempting to answer either of these questions, the student of the teaching of arithmetic should inquire further: (1) What is involved in understanding an

arithmetical fact such as $5 + 4 = 9$? (2) How is arithmetic made meaningful? (3) What is the difference between a teaching procedure that makes for understanding and one that does not?

To provide some background for consideration of the above questions, a part of a second-grade number lesson and a discussion of this lesson is reproduced in the paragraphs that follow.

A Second-Grade Number Lesson

At the beginning of the school year, a new arithmetic program purporting to make for better understanding had been introduced in the Bradford School. In Grades One and Two this new program emphasized the building of number concepts. In order to keep the teachers from hurrying through or slighting the program, good commercial workbooks designed for the building of concepts were supplied.

The principal of the Bradford School was enthusiastic about the concept-building program and had invited the superintendent to visit a second-grade class.

On the day of the superintendent's visit, the second-grade children were considering the following workbook exercise:

Two cakes, one labeled Jane's, the other Bob's, were pictured. On Jane's, there were six candles; on Bob's there were eight. The question "How many candles in all?" was written under the picture.

Since many children had the accepted answer, the teacher wisely directed the discussion to consideration of the methods they used in getting the answer. Among the methods meeting the approval of the teacher were the following: (1) "I counted the candles by twos." (2) "I counted the candles on each cake, and I knew that 6 and 8 are 14." (3) "I counted all the candles." One child offered the following: "I counted the candles on one cake and then counted the candles on the other

cake. I then said there were 6 and 8 candles." The teacher and other children after some discussion decided that this solution, 6 and 8, was not acceptable because it did not tell how many in all.

In an attempt to help those children who were not yet able to visualize quantities, the teacher represented the number of candles on each cake by marks |||||| |||||||| on the board. In dealing with this semi-concrete [1] representation of the quantities, children who needed to count by ones were permitted to do so. The more advanced children again counted by twos, or grouped and added.

As the superintendent and the principal left the classroom, the latter said, "That's my idea of a meaningful approach to arithmetic. Those children are not just learning that six and eight equal fourteen, they are getting the meaning of arithmetic."

"They are getting some valuable experience all right, but I'm not so sure about their getting the meaning of arithmetic," remarked the superintendent. He then asked, "How about that child who said 'six and eight' for the number of candles?"

"Oh, that child is making progress," answered the principal. "It's too bad we can't give him full credit for that answer."

"The truth about that situation," said the other, "is that his statement did answer the question, *How many in all?*"

"What do you mean?" asked the principal.

"Just what I said," replied the superintendent. "Six and eight tell how many candles there were in all just as much as do ten and four."

"I don't follow. What do you mean by ten and four? I thought it was fourteen," was the next comment of the principal. Then he went on, almost talking to himself: "Oh, yes, I guess I see. Fourteen is the same as ten and four. But then why do we accept fourteen as correct and not six and eight?"

[1] Semi-concrete representation refers to the use of pictures or drawings as a substitute for the objects under consideration

"There is a reason, all right, but often our arithmetic books and our arithmetic instruction fail to make that reason clear. I don't blame that teacher for not seeing the point when our program in arithmetic does not even mention it," answered the superintendent.

The reason for accepting *fourteen* as a better answer than *six and eight* was then explained as follows:

"Our number system is constructed on the simple and easily understood principle of grouping by tens. Number designations, such as 14, 17, 26, and 65, together with their significant names, fourteen, seventeen, twenty-six, and sixty-five, make the how-many-in-all concept of totals greater than ten easy to express and to understand. Each number from 1 to 9 inclusive is thought of as representing respectively so many separate *ones* without any suggestion of grouping. When we reach 10, however, we think of that number as standing for a definite collection, a group of ones, and the group, one *ten*, becomes the base of all numbers above ten. Thus, 11 is thought of as 1 *ten* and 1 *one;* 12 as 1 *ten* and 2 *ones;* 13 as 1 *ten* and 3 *ones;* and so on until we pass 19 and reach the second ten, which we write as 20 and which we call twenty, meaning two *tens.* All other numbers to 100 are named as so many *tens* and so many *ones.*

"It is easy to see that there are seven ways to arrange fourteen objects into two groups.

13 and 1	10 and 4	9 and 5
12 and 2		8 and 6
11 and 3		7 and 7

It is also obvious that, to avoid confusion, we must agree on one way of naming the total represented by each of the seven groups. Each grouping gives a different meaning to the total number of objects, but only one grouping, 10 and 4, conforms to the pattern of our number system. Because it conforms, we can write the total with just two digits, 14 (no 'and' required),

and we can speak and write the total with just one word, *fourteen.* No other grouping that represents the total can be expressed so simply in figures or in words. For that reason we are justified in accepting '14' as better and simpler than '6 and 8' as an answer to the question, How many in all?

"Fourteen, then, not only has the advantage of being the rightfully accepted answer, but it is also simpler to think with than is six and eight. Simplicity, however, depends very much on familiarity which comes from experience. Before fourteen can be considered simpler for the child, one must be sure that the child has had enough experience with fourteen as a ten and four ones to enable him to see that that way of considering the quantity is simpler than thinking of it as one group of six ones and another group of eight ones.

"From that brief part of the lesson we saw, there is no indication that the children in this second grade were thinking of 14 as 1 ten and 4 ones or whether they were thinking of 14 as 14 ones. The manner in which the marks were used seems to indicate that it was the purpose of the teacher to have the children think of the answer as fourteen ones. As was suggested earlier, the child can be correct, as far as his representation of quantity is concerned, if he thinks of fourteen as fourteen ones. In that case, however, if he is to get understanding of the quantity involved, either he has to break the fourteen into smaller groups or build up a sort of pattern or model for his idea of fourteen. If merely breaking the whole number into smaller groups is to be the method of getting understanding of quantity, the use of 6 and 8 is quite proper. But a regrouping of this sort entirely ignores the number system as such Because ten is the base of our number system, it is a most useful group, and our arithmetic program will serve the child's need best if it directs his learning toward the use of the 'ten' idea. Upon that idea our number system is built."

After listening to the explanation, the principal remarked:

"I begin to see that there is much more to this 'meaning theory' than I thought. We thought we were helping children see sense in what they are doing by having them work from pictures and drawings to the answer, by having them count by ones and by twos, and by having them say 6 and 8 are 14. In the light of what you have said, those procedures don't necessarily make for meaning or understanding. It seems to me that we need to understand numbers better ourselves before we talk about teaching them in a meaningful way."

The incident described in the preceding paragraphs might have taken place in any one of hundreds of elementary schools. Instructional procedures in use are seldom examined critically by either the teaching or the administrative staff; and even if a critical examination were undertaken by a person such as the principal of the Bradford School, little would be gained. From the incident described it might also be concluded that theories of instruction are not clearly understood by those who are trying to put the theories into practice.

Theories of Arithmetical Instruction

From the discussion between the two observers it is evident that the new number program was an attempt to put a "meaning theory" of arithmetical instruction into practice. It may also be inferred that this "meaning theory" was comparatively recent and that its application to practice was not understood very well by the principal. A clearer picture of the issues involved in the discussion of the second-grade number lesson will be obtained if the significant features of the "meaning theory" as well as those of the other common theories of arithmetical instruction are understood.

At present, there are three theories of instruction which are usually considered. These are the "drill theory," the "theory of incidental learning," and the "meaning theory." While none of these is a theory of instruction in the usual sense of the

word, it can be said that arithmetical instruction has been influenced through thought stimulated by these three so-called theories; and as the National Council Committee on Arithmetic [1] has stated, it is convenient to refer to them as theories. In the next paragraphs, the main characteristics of each of the three theories are presented.

The drill theory is the oldest of the three, and, if textbooks are indicators of teaching practices, it is by far the most widely applied. Advocates of the drill theory of instruction maintain that the facts and processes of arithmetic are most easily learned by repetition — that is, by saying or thinking the fact over and over or doing the required process again and again. For example, the theory holds that by thinking or saying a fact (such as $6 \times 4 = 24$) a number of times, that fact will be learned. In the same way, it holds that the process of borrowing in subtraction is learned by solving many examples in which the borrowing process occurs. But this repetition is not to be devoid of thought or attention. The fact or process being learned must be understood by the learner. Initial instruction, therefore, requires a detailed demonstration of the truth of each fact and a complete explanation of each process. After instruction, drill is used to assure mastery. To avoid monotony, there are provided drills, tests, number games, work sheets, and problems to give practice in the facts and processes being taught. The drill theory also focuses instruction and practice upon discrete or isolated phases of arithmetic. For example, a fact such as $6 \times 8 = 48$ is learned without any reference to a closely related fact such as $5 \times 8 = 40$. The drill theory is further characterized by the great reliance placed on thorough mastery of the material studied.

The incidental-learning theory, although fairly well known for at least twenty years, has never been widely accepted in

[1] R. L. Morton, "The National Council Committee on Arithmetic," *Mathematics Teacher*, October, 1938.

either theory or practice. This theory of instruction states that arithmetic can be most effectively taught if instruction is undertaken only when a child has a need for a fact or a process. To illustrate: A child will not be required to learn multiplication with a two-figure multiplier until some problem in his life, either in school or out of school, calls for the process. Proponents of the theory contend that the need or use that the child has for facts and processes will insure both understanding and retention of the things learned, and for this reason they deny the necessity of planned repetition which is considered so essential in the drill theory. According to the incidental-learning theory, there should be no systematic, logical step-by-step teaching of arithmetic. The content of arithmetic taught would depend upon the occurrence of numbers in the other school activities of the children. By being concerned only with arithmetic for which the children have already experienced a need, the advocates of the incidental-learning theory contend that both meaning and understanding are assured. In actual practice the theory has seldom worked, but it has influenced teaching considerably. Note, for example, the attempts of textbook-writers to have problems and other number situations grow out of some activity in which the child might participate.

The meaning theory is comparatively new in name, having received its first major publicity when it was presented in the *Tenth Yearbook* of the National Council of Teachers of Mathematics in 1935.[1] As indicated by its name, this theory places more emphasis on meaning than do the two theories already described. In fact, this emphasis on meaning is the outstanding characteristic of the theory. According to the National Council Committee on Arithmetic the theory implies

a kind of arithmetic in which both the mathematical and the social aims are clearly recognized — and clearly recognized as

[1] *Tenth Yearbook*, National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1935), p. 19.

interdependent and mutually related. Attainment of the mathematical aim is regarded as possible only if meaning, the fact that children shall see sense in what they learn, is made the central issue in arithmetic construction. Arithmetic is conceived as a closely knit system of understandable ideas, principles, and processes, and an important test of arithmetical learning is an intelligent grasp upon number relations together with the ability to deal with arithmetical situations with proper comprehension of their mathematical significance.[1]

The meaning theory, then, is characterized by the viewpoint that arithmetic can be learned most easily *if children see sense in what they do and if arithmetic is taught as a closely knit system of related ideas, facts, and principles.* The theory in its present form is so recent and is interpreted in so many different ways that no one description would do justice to the arithmetic procedures based on it. According to Thiele,[2] great reliance would be placed upon the child's *discovering* for himself effective solutions and upon his seeing relationships. According to Buswell, Brownell, and John,[3] much use would be made of directing the children toward the learning of the particular process or fact through performance of a carefully planned series of steps involving concrete experiences followed by the identification of certain characteristics. According to Knight,[4] emphasis would be upon the number of situations that arise in the room, such as "How many books do we need?" The solution to these number experiences would then be carefully explained and the difficult or tricky procedures duly emphasized.

[1] R L Morton, "The National Council Committee on Arithmetic," *Mathematics Teacher*, October, 1938.

[2] C. L. Thiele, in *Sixteenth Yearbook*, National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1941), pp. 46, 53, 55–57.

[3] Buswell, Brownell, and John, *Daily Life Arithmetic* (Boston. Ginn and Company, 1938) See book for Grade Six, pp. 47–58

[4] F B. Knight, "Providing Meaningful Number Experiences," *Primary Activities* (Chicago: Scott, Foresman and Company), February 15, 1941.

The interpretations of many other authorities could be cited, since practically everyone accepts verbally the meaning theory. The preliminary report of the National Council Committee on Arithmetic was published in three leading periodicals and criticism was solicited. Even after two years not one person had disagreed in published writing with the viewpoint assumed. The major effects that this theory has had on teaching are that it has (1) increased the emphasis given to concept-building; (2) increased the use of concrete and semi-concrete materials; (3) stimulated the recognition of the value of relationships in arithmetic; (4) stimulated attempts to teach the system of number rather than separate elements of knowledge; and (5) emphasized having children see reasons for the work they do in arithmetic. Of course, the aim of these procedures is to make for better understanding. The proponents of the theory have assumed that the more clearly and more thoroughly the pupil understands facts and processes, the better arithmetician he will become.

This current emphasis upon meaning is by no means new. At various times in the past, influential leaders have campaigned for exactly this same thing. The committee which prepared Bulletin Number 13 of the United States Bureau of Education, *Mathematics in the Elementary Schools of the United States* (1911), stated very definitely in the discussion of method that some teachers put great stress on meaning. In the foregoing presentation of the theories of arithmetical instruction, two points stand out. First, although one of the theories emphasizes understanding more than the others, all the theories give understanding an important place in the learning of arithmetic. Second, the methods of attaining understanding vary markedly. In the drill theory, understanding is supposed to be secured through careful explanation during initial instruction. In the incidental-learning theory, understanding is supposed to be attained when first study is undertaken at the time the learner

has a need for the fact or process he is studying. In the meaning theory, understanding is supposed to be secured by letting the learner see the reason for studying the fact or process and by emphasizing the relationships between various aspects of the number system. The important conclusion to draw from the discussion of the theories of instruction is the fact that helping pupils to understanding is one of the major concerns of those who are interested in the teaching of arithmetic. Why this emphasis on understanding? The best answer is that many children who study arithmetic can perform operations, but they do not understand the operations they use.

The issues raised in the foregoing discussion of the second-grade number lesson show clearly that arithmetic is not very well understood. That such a condition can exist, when all recent textbooks profess to be built to develop *meaning* and *understanding*, should indicate to the student of the teaching of arithmetic that something more than just talking or reading about the number system, or about making numbers meaningful, is essential to an adequate instructional program.

Why Arithmetic Is Not Understood

In the last paragraph of the preceding section the inference was made that not all teachers understand arithmetic very well. That the teacher must understand the number system and see the relationships to be emphasized seems clear, for not until he has such knowledge will an adequate teaching program be attained.

The principles of arithmetic are poorly understood, not because these principles are inherently difficult, but because they have not been properly taught. The primary reason for this poor teaching is, of course, the fact that most of those who have taught arithmetic in the elementary school have not themselves understood the principles of the subject. Before interpreting the last statement as a criticism of former teachers,

consider the fact that these teachers probably never had an opportunity to learn the principles. Furthermore, much of arithmetic can be used in a limited way with only a superficial knowledge of how or why it operates as it does. For example, you do not need to know why you carry in column addition in order to use the carrying process. Then, too, textbooks in arithmetic have done little to develop a better understanding of the subject. Yet the blame does not rest even there.

Through the hundreds of years that arithmetic has been used, some of the underlying principles have from time to time been neglected or forgotten. As a result of this neglect, arithmetic has often been used without understanding, just as machines are frequently operated by those who do not understand the principles on which they operate. Often the operator of a machine may secure efficient results without understanding how the machine works, but the operator of arithmetic who attempts to use arithmetic without understanding its principles gets absurd and incorrect results and sometimes fails completely to make full and reliable use of a valuable tool. Many people, therefore, use number at a low rate of efficiency. They possess a verbal knowledge of terms and processes, but instead of being confident in the practical application of this knowledge of numbers, they are so beset with uncertainties that they flounder about, mathematically, and are frequently unable to solve their problems at all.

For the most part, arithmetic has been taught by people who have a poor understanding of number and its uses. The results of such teaching have already been indicated. Most children, even after intense effort on their part and hard driving by the teachers, get only a superficial mastery of arithmetic. Of course, some children attain a high level of efficiency in the use and understanding of numbers in spite of the teaching they receive. But because of the lack of understanding by teachers and writers of textbooks, arithmetic has become a difficult and

often almost a useless study. The lack of understanding exhibited in the second-grade lesson described in the first part of this chapter is relatively minor in comparison with some examples that might have been chosen. With a little help, however, that second-grade teacher could probably make arithmetic much more meaningful and certainly more useful to the children.

This discussion of why arithmetic is not understood is intended to emphasize the importance of a thorough knowledge of the principles upon which the content of arithmetic is based. In the next section some of the outstanding principles are briefly described. Throughout the remainder of the book these principles will be elaborated and the teaching procedures illustrating them will be presented.

Meaning and Understanding

Meaning, according to the National Council Committee on Arithmetic, is characterized by the child's seeing sense in what he does. Applied to the second-grade number lesson described earlier in this chapter (pages 2–6), this definition would require that the exercises be such that the child is able to see why 14 is the best answer for "how many in all." While there are many ways to give children an opportunity to see the reason for using 14 rather than 6 and 8, or fourteen ones, only one way of presenting the advantage to children is offered here. Suppose, for example, that the task had been to report through symbols, to an absent classmate, that fourteen important things were new to the room, six added on Monday and eight on Tuesday. In such a situation the child could certainly see, in economy of effort alone, a reason for using the short, commonly accepted symbol "14" rather than fourteen separate marks ||||||||||||||, or two numbers and a word, "6 and 8." Meaning, then, involves seeing not only that a number statement is true or that a procedure is correct, but also why one way of

making the statement or of doing the process is better than all other ways.

The meaning theory, as defined by the National Council Committee on Arithmetic, places heavy emphasis on the understanding of number and arithmetical processes and upon the teaching of number as a closely knit system. The theory has been widely accepted verbally, but, as was shown in the discussion of the second-grade lesson, it may not be understood adequately. It is doubtful whether many of those who are trying to apply the theory have asked themselves the exact meaning of terms like "understanding" and "system." In order that a common ground for use of such terms may be available, a brief discussion of them is included here.

It is easy to use the word "meaning," or "to get the meaning of," for "understanding"; but just what does the child possess who has a good understanding of a number such as 7? Among all the other facts which might be known, for example, he knows that 7 comes after 6 and before 8; that 7 is 2 more than 5, 2 less than 9; that it is 41 less than 48; that it is the sum of 4 and 3, or of 6 and 1, or of 3, 2, and 2; and that it is one-half of 14. Analysis of what this child knows reveals that every part of his understanding of 7 is in some way an expression of the relationship that exists between 7 and other numbers. The understanding of a number, then, is the seeing of its relationships to other numbers. Since every fact or concept has an infinite number of relationships, no one can ever have complete understanding of it in terms simply of relationships. There are, however, as consideration of the above will show, some relationships that are far more significant than others. It is the job of the arithmetic program to give children an opportunity to learn these more significant relationships.

In brief, then, meaning and understanding are defined as follows: Meaning is the seeing of reasons for, the import of, or the sense of, a process. Understanding is the seeing of relationships between a fact or process and other facts or processes.

Our Number System

The expressions *number system* and *teach number as a system,* like the term *understanding,* are frequently used with only a vague idea of what is implied by the term *system.* Many points which characterize numbers as a system can be listed. The outstanding characteristics are the following:

1. The Hindu-Arabic numerals, if used in the *ordinal* sense, always have the same order, 5 always comes after 4, 7 after 6, and so on. This order holds regardless of the size of the quantities (units, tens, hundreds, or millions) referred to. It is this ordinal or positional aspect of number that gives us our most usable concept of quantities like 9, 12, or 52.[1] To illustrate this point, think of the number 52. What devices do you use to get an idea of a quantity 52? Those who use numbers most efficiently would think of 52 as about one-half of a hundred; a little more than the fifth group of ten. In both of these ways of getting at the meaning of 52, the position of number in a series plays the major role.

2. The first nine numerals represent quantities thought of as groups of ones or units. After 9, the next number, 10, is not always efficiently thought of as representing individual entities, but is thought of as a single entity or collection. This fact is shown even in the name ten, where the familiar terminology of 1 is implied, though not actually used by most people, for this number is the first ten or one ten. This use of collections of ten, with repetitions of the ten as larger quantities are encountered, is what makes our number system a decimal system. It has a base of ten. The base, then, is nothing more than the first standard or base collection that is used. If this collection idea is properly utilized, 12 is not twelve separate entities; it is three entities consisting of a collection of one ten and two ones. Forty is four entities or four collections· it is four tens. To demonstrate to yourself the simplification, represent each of the

[1] D. E. Smith, *The Teaching of Elementary Mathematics* (New York: The Macmillan Company, 1921), p 103

preceding numbers first as ones and then as collections of tens plus the necessary ones.

3. The names for the tens consist of terms that are closely related to their meaning. Twenty can be easily related to *twin tens;* thirty is the third ten. Note that the suffix *ty* in all such names as twenty and thirty means tens. Thus, in dealing with the tens (twenty, thirty, forty, and so on) a person does not learn entirely new names with a special place in the series. For the most part, the names in the *tens* series are much like those of the first nine numbers: the first syllable of *twenty, thirty, forty,* and *fifty* suggests *two, three, four,* and *five* respectively; and in the terms *sixty, seventy, eighty,* and *ninety,* the exact number names for the ones constitute the first part of each of the number names for the tens. The order of the tens is exactly the same as that of the numbers from 1 to 9.

4. The use of place value simplifies the writing of numbers. By place value is meant the fact that the value of a digit is dependent on its position in a number. As an illustration, consider the numbers 3, 35, and 320. In each of these numbers, the numeral 3 is used, but in each number it has a different value because of its position. This principle of place value combined with the use of a placeholder (zero) makes possible the writing of any number through the use of only ten different symbols.

System, then, as applied to number implies exactly what the term denotes in other fields; namely, order and method An arrangement that makes possible the operation or understanding of a process, with a minimum of effort, characterizes any system. Close study of the four characteristics of our number system just listed will show how the system makes for the conservation of effort. One example of the conservation of effort for each characteristic is offered here: (1) Since the number names always have the same order, learning the relation that five thousand has to six thousand presents little difficulty. (2) The collection idea makes the addition of two thousand and

four thousand as easy as the addition of two and four. (3) In learning to count to one hundred, the similarity of the tens' names to the ones' names makes the task of the learner comparatively easy. (4) The economy which the fourth characteristic makes possible is clearly seen by comparing the Roman and the present system of writing the year nineteen hundred forty-nine: MCMXLIX and 1949. System is important in all work, because it makes for economy of effort and therefore results in efficient usage.

The discussion of definitions of meaning, understanding, and number system should emphasize further to the student of the teaching of arithmetic the importance of seeing the full import of terms and phrases used in discussions of teaching procedures. Knowing a definition of a term is in itself of little value. The student of the teaching of arithmetic must be able to illustrate definitions by reference to classroom procedures, to descriptions of knowledges or achievements of children, or in other ways apply the definitions to the teaching situation. Otherwise, the best terms and phrases and their definitions may become just so many high-sounding words and thereby become a handicap rather than a help to teachers. "What does that mean in terms of classroom procedure?" should, then, be one question used in the consideration of all terms and principles of arithmetic teaching.

STUDY QUESTIONS

Directions: From the several responses suggested, choose the one which you consider the most acceptable answer to the question. An "N" means that none of the given choices is acceptable to you. In order to facilitate further use of the study questions, it is a good plan to record your answers on a separate sheet of paper. Try all ten questions without referring to the text.

1. Why is 10 and 2 a better answer for the question "How many are 7 and 5?" than is 6 and 6, 8 and 4, or 9 and 3? (*1*) It tells how many in all. (*2*) 10 and 2 is more easily thought of as

one number than is any of the other combinations. *(3)* 10 and 2 is the accepted pattern. *(4)* N.

2 Of the many relationships about the number 5 that a child learns, which of the following is the most significant? *(1)* That it comes between 4 and 6 *(2)* That it is the equivalent of 3 and 2. *(3)* That it stands for 5 separate ones. *(4)* That it is one more than 4.

3. Why do the numbers thirty, forty, and the like all end with "ty"? *(1)* To make them rhyme. *(2)* To distinguish them from the tens and ones numbers. *(3)* There is no good reason *(4)* N.

4. What is the base of a number system? *(1)* The tenth or last numeral. *(2)* The positional value scheme of writing numbers. *(3)* A ten. *(4)* N.

5. What is meant by place value as applied to our number system? *(1)* That our numbers when used in counting always have the same place or order. *(2)* That the value of a number is determined by its position. *(3)* That hundreds numbers have the same order as ones numbers. *(4)* N.

6. About how much of the average elementary-school child's time is spent in the study of arithmetic? *(1)* 3–5%. *(2)* 6–8%. *(3)* 10–12%. *(4)* 14–16%.

7. Which of the three common theories of arithmetical instruction gives little or no emphasis to understanding? *(1)* The drill theory. *(2)* The incidental-learning theory. *(3)* The meaning theory. *(4)* N.

8. In order to use an arithmetical process, such as borrowing in subtraction, is understanding of that process essential? *(1)* Yes. *(2)* No.

9. Does the person who has a good understanding of numbers ever think of such numbers as 13 as so many separate and distinct ones or does he always think of 13 as a ten and three ones? *(1)* He always thinks of the ten and three ones. *(2)* He sometimes thinks of thirteen ones.

10. Which of the four major characteristics of our number system is probably the least important? (*1*) The one referring to order. (*2*) The one referring to the collection idea. (*3*) The one referring to the similarity of names for ones and tens. (*4*) The one referring to place value.

Now check your answers to the questions by referring to your text. Comparison and discussion of your selections with others are excellent ways to secure better mastery of issues and facts to which the study questions refer.

2

Purposes, Principles, and Methods of Instruction

Accepted Views of the Place of Arithmetic

In the preceding chapter some of the outstanding features of arithmetical instruction which put emphasis on meaning and understanding were introduced. Before beginning the extensive study of such an arithmetic program, the student should know something of the place of arithmetical instruction in the school program, the objectives of instruction, the principles of learning, and the most common methods of instruction. These and related topics are considered in the present chapter.

Arithmetic as an elementary-school subject has enjoyed and still holds an important place in American schools. According to a recent survey, about ten per cent of every elementary-school child's time is devoted to direct study of the subject. While this amount of time is considerably less than that formerly given to the subject, it is still a large percentage. Few other elementary-school subjects take as much of the total time as is given to arithmetic.

The fact that arithmetic has long been an important school subject is of itself little evidence of its intrinsic value. The value of any subject must ultimately rest upon the service that

it renders to those who study it. Arithmetic is the study of numbers, their numeration and notation, the processes whereby they are handled most economically, and their uses in everyday life where quantities are involved. That these various facets of arithmetical study are of service to man appears so obvious that it hardly seems worth while to mention instances here. For sake of emphasis, however, one example of the ordinary uses of each aspect of arithmetic will be listed.

Whether it be in determining the hour of quitting work, in recording the number on an insurance policy, in reading the distance indicated on some road marker, or in doing some other common task, numeration and notation are used almost daily by every person. In fact, little reading or writing occurs without the reading and writing of a few numbers.

The processes of addition, subtraction, multiplication, and division are used in connection with many problems of daily life. Even though many people use pencil and paper infrequently in performing their calculations, they often employ the processes mentally to simplify quantitative situations, such as checking change to be received, trying to determine whether or not there is sufficient time to warrant the undertaking of certain activities, and estimating quantities like the number of cattle in a herd. In addition to the use of arithmetical processes for simplifying situations, there are many other uses of number in life. For example, numbers are employed in transmitting news, in reading books, and in presenting social problems and scientific problems which require the expression of definite quantities.

The role of numbers in thinking is one of simplification. Consider the following examples: (1) Suppose that you have assigned for tomorrow three unrelated tasks. How much that little concept of *three* helps you to remember the tasks! (2) Assume that you see the following legend under a picture, "This largest open pit mine on the Mesabi Range, which is $2\frac{1}{2}$ miles long, one mile wide, and 350 feet deep, contains 70 miles of railroad track-

age." Notice how you use the numbers in thinking of the size of this mine. (3) Suppose that you read in a magazine the following. "In 1918 Ekaterinburg, a town just east of the Urals, had a population of 70,000 people. Twenty-five years later, the same town, called Sverdlovsk, had 500,000 people." Didn't the numbers help you get an idea of the phenomenal growth of this Russian city? (4) Assume that you hear a news reporter announce. "The tremendous saving in weight by dehydration of foods is shown by the fact that 18 cases of eggs when dried weigh only 175 pounds." This last example illustrates the need for number knowledge other than that given in the statement. Such situations frequently confront us and show in a dramatic manner how ineffectual our thinking on some problems must become when numbers are omitted. Each of the four examples shows how numbers make for economy of thought. They help to simplify the concepts. Considered in this sense, numbers become essential tools of language and thought, and the value of arithmetic is then almost self-evident.

An additional service that arithmetic can offer to education comes through the study of the history of number. The historical aspects of arithmetic are an important part of our social heritage and are undoubtedly valuable to the cultural development of children. The fact should be noted, however, that historical phases of arithmetic are used very little at the present time in the classroom.

In the past the purposes of teaching arithmetic have not often been clearly stated, but from the literature of various times it seems reasonably certain that purposes have varied from "narrow utilitarian" to equally "narrow disciplinarian" ones. By narrow utilitarian is meant the teaching of only specialized business or commercial arithmetic. The narrow disciplinarian purpose implies the use of arithmetic primarily as a subject by means of which the mind is trained. The methods used in teaching were, of course, greatly influenced by the purpose for

which society had permitted arithmetic to be taught in the school. If the aim was the preparation of accountants, clerks, and so forth, then heavy emphasis was placed on the business procedures. If the aim was mental discipline, then difficult exercises were included in the teaching without regard to the possible application of such exercises to life outside the school.

It is doubtful whether instruction in any given school represented only one or the other of the two purposes identified. It is certain, however, that the two purposes described, along with many others, affected the arithmetic curriculum. For further discussion of the historical purposes of arithmetic, the reader is referred to studies on that phase of the subject.[1]

Proposed Purposes of Arithmetic

Since content and method are affected to a marked degree by the purposes for which a subject is being taught, the purposes or objectives of the teaching of arithmetic merit careful consideration. In a teaching program which gives emphasis to understanding, this consideration becomes doubly important.

The purposes of arithmetic upon which the program of teaching outlined in this book is based are given in the four numbered statements below:

1. To develop methods of exact thinking in situations in which a consideration of quantity is essential. Three levels or degrees of quantitative thinking are illustrated in the statements that follow: (*a*) "During recent times some of the original soil of our cultivated slopes has washed away." (*b*) "During the last five years two inches of our original soil has washed away." (*c*) "During the last five years about one-fourth of the eight inches of the original soil has washed away."

[1] W. S. Monroe, *Development of Arithmetic as a School Subject*, Bureau of Education Bulletin No 10 (Washington, D C · Government Printing Office, 1917). Florence Yeldham, *The Teaching of Arithmetic Through Four Hundred Years* (London: Harrap and Company, Ltd , 1936).

Although these three statements cannot be called exact illustrations of thinking, they are representative of ideas or facts that people marshal when confronted with such a problem as soil erosion.

The first illustration represents the lowest or least precise level of thinking. At least, as far as communication of ideas is concerned, no data are given which would permit the reader to know whether the loss was just a trace, an appreciable amount, or a major portion of the soil. The second statement gives the amount of loss, and if the person making the statement knows the significance of that amount, good thinking can be done. In the third statement more facts have been marshaled and their interrelationships have been noted. The idea is now in the most usable form for getting an understanding of the significance of the washing away of the soil.

2. To provide a vehicle (tool) for establishing order, system, and punctuality.

"Seat 15, row 55," "One out of every 20 is left for seed," and "Turn to page 644" are examples of how numbers are used to establish order.

"To fill an order for 2000 sheets of paper the clerk takes 4 packages of 500 sheets each" is an example of how numbers are used to systematize work.

"The appointment was made for 10 : 30 A.M., September 18, 1946," is an example of how numbers make it possible for man to be punctual.

3. To provide pupils with enough knowledge of mathematical processes and business procedures to enable them to solve efficiently the ordinary quantitative problems of everyday life.

"To find the cost of draperies, Mary multiplied $2.36 (the cost per yard) by 16 (the number of yards needed)." This economical way of finding costs illustrates the third objective.

4. To furnish knowledge of the development of numbers and of weights and measures as a basis for better understanding of

civilization. Learning how ancient engineers and clerks computed with an abacus, thereby avoiding the difficulties inherent in their clumsy notational system, represents the type of knowledge referred to in the fourth objective.

The major contributions of arithmetic are implied in the first purpose cited. Unless an individual has learned to use numbers to aid his thinking in terms of quantity, his thinking must of necessity remain slow and ineffectual in such situations. Consider again the three levels of thinking given to illustrate the first purpose. In the highest level ("During the last five years about one-fourth of the eight inches of the original soil has washed away") not only have numbers been used to give precise statements of amounts, but the relationships of amounts have been discovered and again expressed in exact language. When such language deals with quantities, number is an indispensable tool. By simplifying or clarifying quantitative concepts, arithmetic does its greatest service. If the mind is relieved of cumbersome ways of thinking about concepts, then it is in a position to do something with the concepts Thus numbers become an instrument of thought The purpose of developing methods of exact thinking is so important that no other objective or purpose of arithmetic is necessary to justify the time given to it in school.

Further attention is called to the second purpose because it is so frequently neglected in modern arithmetic programs. This neglect is primarily due to the fact that the uses with which this phase of number deals are taken for granted. It is generally assumed that anyone readily sees how number is used to indicate order, secure system, and achieve punctuality. To adults these uses appear very simple, but to the small child such concepts may have little meaning. This second purpose or objective should receive, therefore, a great deal of emphasis in the primary number program.

The third purpose, that of solving the everyday problems of life involving quantity, is so universally recognized that it requires little discussion. The purchase of goods and services and entertainment is engaged in by almost everyone in modern society. On rather infrequent occasions the solution of problems involving quantity becomes very crucial. A few examples are: (1) finding the number of cubic feet of air space in a building; (2) determining the total of all monthly payments on a house; (3) figuring exactly the changes in dimensions required to make a garment three-fourths as large as the pattern calls for. Add to the three preceding problem situations the extensive list of business enterprises in which the solution of arithmetical problems is all-important, and the case for problem-solving appears very impressive. For the best interest of arithmetic, however, some limitations concerning the importance of problem-solving should be noted. In the first place, it should be recognized that the arithmetic of ordinary daily or weekly purchasing is learned and performed by those who have never attended school. Furthermore, the arithmetic of so-called crucial problems is often done free of charge by interested agencies, and only a relatively small percentage of the arithmetic taught in school is used by any one business enterprise. Problem-solving is not, then, as essential to the making of a living as it seems when the various quantitative problems of life are enumerated. We should recognize that success in life for the great majority is not dependent upon their ability to solve number problems.

The preceding statement about the limited use of problem-solving in making a living is not intended to promote the idea that problem-solving is not important. The aim is to recognize the major values of arithmetic as major values and to recognize secondary values as only secondary. Because of unwarranted assumptions made by uncritical persons, problem-solving is in present practice often given more emphasis than it deserves.

PRINCIPLES GOVERNING THE SELECTION OF LEARNING EXPERIENCES

The major task of the arithmetic program, then, is to provide learning experiences which will give the child an opportunity to attain the four major purposes or objectives listed in the preceding section. And the first and most important principle governing the selection of learning experiences is that children must be given the opportunity to learn through experience. Probably because of its very simplicity and because other principles of teaching seem so complex, this axiom of teaching can be neglected easily. Students of the teaching of arithmetic, however, should understand that this first principle is directed at the problem of securing in any given instructional activity a prominent place for the child's achievement of meaning.

Consider, for example, the fact that present-day arithmetic is rightly concerned with teaching the best methods of carrying on such processes as adding, subtracting, multiplying, and dividing. If the first principle is applied in its fullest sense to that teaching, the child must have experiences which will enable him to see that these methods *are* the best methods. Likewise, if the child is to learn that numbers make for exactness and clarity of thought, he must have experiences which demonstrate this exactness and clarity.

As defined in Chapter 1, understanding is the seeing of relationships among the various items that constitute our number system. It follows, therefore, that the nature of the number system will affect markedly the selection of arithmetical content. This basing of the selection of part of the content of arithmetic on the nature of the number system is the second principle to be used in the selection of learning experiences.

For the sake of brevity, a third principle governing the selection of learning experiences may be called the *principle of familiarity*. This principle states that familiar situations shall be

used for the settings in which are presented the phases of number to be taught. Familiar situations are recommended because they offer the maximum possibilities for the child to gain understanding.

A fourth principle recognizes that each number fact has many significant relationships with other number facts. For example, the sum of $5 + 3$ is the same as the last number used when a group of 5 objects and a group of 3 objects are counted as a single group; the sum is the same as the sum of $4 + 4$, $6 + 2$, $3 + 5$; the sum is one more than the sum of $4 + 3$, $5 + 2$. For efficient learning of these many relationships, a relatively long period of instruction providing various experiences is essential for the teaching of each number fact.

A fifth principle states that generalizations grow out of experiences. That is, the child must have the experiences out of which he can construct the generalizations.

While there are other principles of selecting learning experiences in arithmetic, the five just listed, plus the statement of purposes given previously, furnish a good foundation for the study of the following discussion of methods of learning.

Methods of Learning

The purposes and principles of arithmetical instruction become significant when translated into actual teaching procedures. By examining these procedures the student of the teaching of arithmetic can see how and to what extent the principles are applied To permit such an examination, outlines of samples of three methods of learning used in teaching multiplication are given on the pages immediately following. To keep these outlines from becoming too lengthy, attention is centered on initial instruction in multiplication with special attention given to the fact $4 \times 2 = 8$. The first and the second samples are very similar to the procedures employed in two widely used textbooks. The third sample is an illustration of the procedures in the

instructional program advocated by the author of this book. Since a single fact, such as $4 \times 2 = 8$, is never taught in isolation, the outlines may appear unnatural; nevertheless, the major steps in each teaching program are presented.

Outline of Sample Program I

A. *Introduction to multiplication situations*

1. Picture of a man making toy two-wheeled and four-wheeled carts. Two are already made. The activity pictured is presented in a child setting and these questions are asked:
   "How many wheels were used to make one cart?"
   "How many wheels were used to make two carts?"
   (The pupil can consult the picture in answering questions.)
2. Two threes, two fours, two fives, four twos, and five twos are pictured in a similar manner and questions similar to the above are asked.

B. *Developing the meaning of two times a number*

1. Blocks are used in nine pictures, each picture representing a different quantity (1 to 9), and each labeled to furnish the setting for nine exercises of this type:
   "How many blocks were used to make one chair?"
   "How many blocks were used to make two chairs?"
   (Picture shows only one chair.)
   3 and 3 are _____. 2 threes are _____.
2. Give the answer to each of these examples
   2 fives are _____. 2 ones are _____. 2 fours are ____, etc.

C. *Presenting the meaning of multiplication and learning multiplication facts*

1. A situation of this type
   "Bill carried 4 chairs to the porch. Then he carried 4 more chairs to the porch. Four chairs were

carried 2 times. How many chairs in all were carried?"

These are two ways to find the answer. 2 fours are 8, or 2 times 4 is 8.

2. 2 threes are 6. 2 times 3 is -----.
   2 sixes are 12. 2 times 6 is -----.
3. Learn these examples and their answers:
   2 times 6 is 12. 2 times 2 is 4, etc.
4. When you say 2 times 6, you multiply 6 by 2.
5. Give the answer to each of these: 2 times 4 is -----. 2 times 2 is -----, etc.

D. Using the multiplication sign

1. What is the sign called that tells you to add?
   What is the sign called that tells you to subtract?
   This sign $\times$ tells you to multiply. When it is read you say "times." The example $2 \times 4 = 8$ is read "two times four is 8," or "two times four equals 8."
2. Watch the signs in reading these examples:
   $5 + 4 = 9$ $2 \times 4 = 8$ $5 - 3 = 2$
3. Give the answers for these:
   $2 \times 5 =$ $2 \times 4 =$ $5 - 2 =$ etc.

E. Using multiplication in solving problems

Multiply to get the answers:

Problem 1. Bob and Tom fed the squirrel. Each boy gave the squirrel 4 pecans. How many pecans did the two boys feed to the squirrel? $2 \times 4 =$

F. Demonstrating the equality of product resulting from interchange of multiplier and multiplicand

$$(2 \times 4 = 8; 4 \times 2 = 8)$$

1. Pictures showing two fours and four twos furnish the setting for such questions as "How many were used for one? How many for two?" and "How many for one? How many for four?" Following the first pair of questions appear, "2 fours are -----. 2 times 4 is -----, and following the second pair, 4 twos are -----. 4 times 2 is -----."

2. Two fours are the equal of four twos.
   2 times 4 = 8. 4 times 2 = 8.
3. $2 \times 4 = 8$ $4 \times 2 = 8$
   $2 \times 6 = 12$ $6 \times 2 = 12$ etc.

G. Practice exercises

1. Read and give the answers:
   $4 \times 2 =$ $2 \times 3 =$ etc.
2. $2 \times 4 = 8$ and $\begin{array}{r} 4 \\ \times 2 \\ \hline 8 \end{array}$ are read the same way.
3. Read and give the answers:
   $\begin{array}{r} 4 \\ \times 2 \\ \hline \end{array}$ $\begin{array}{r} 2 \\ \times 3 \\ \hline \end{array}$ $\begin{array}{r} 2 \\ \times 4 \\ \hline \end{array}$ etc.

OUTLINE OF SAMPLE PROGRAM II

A. Introduction to multiplication situations

1. A picture of eight apples placed by twos provides the setting for the question, "How many apples on the table?" Then follow these suggestions: "There are 2 apples in each group, so you add by 2's: $2+2+2+2 =$ _?_; or count by 2's · 2, 4, _?_, _?_." The answer and multiplication fact used are then given. "There are 8 apples. Four 2's = 8."
2. Another multiplication fact is introduced in a similar manner and then four 2's are pictured again and both four 2's and two 4's are developed through use of the pictures and the incomplete statements, Four 2's = _?_ and two 4's = _?_.
3. The next procedure presents problems involving multiplication facts for 2's and 3's. Each problem is followed by such a statement as "Four 2's = _?_, and for the first problem the pupil is told that the answer can be found by adding, by counting by 2's, or by drawing dots. The pupil is directed to use the drawing of dots if he cannot add or count by 2's or 3's.

B. Developing the meaning of multiplication

1. Pictures of horse teams at a fair were the setting used for the following:

   Tom said, "1, 2, 3, 4, 5, 6. There are six horses to that wagon."

   "Why didn't you count by 2's?" asked Henry. "2, 4, 6, that's easier."

   "I know a still easier way," said father. "I say three 2's are 6. That is multiplying."

2. Multiplication as a short way of adding is then discussed. As a ten-horse team appeared, father said, "How many horses?"

   Tom counted 2, 4, 6, 8, 10. Henry added five twos. Do five twos equal 10? Then, pointing to an eight-horse team, father said, "How many horses to this wagon?" What was the boys' answer? Four 2's = -----.

3. Find the answer by adding 2's:

Four 2's = -----	2
	2
Six 2's = -----	2
	2
Etc.	8

4. Make multiplication problems with these mittens. (Eight pairs of mittens are pictured.)

C. Writing and reading multiplication facts

1. You can write the multiplication facts three ways.

   (*a*) Four 2's = 8 (*b*) $4 \times 2 = 8$ (*c*) $\begin{array}{r} 2 \\ \times\ 4 \\ \hline 8 \end{array}$

   You read them all the same way: "Four twos are eight" or "Four twos make eight."

2. The sign $\times$ is the multiplication sign. It tells you to multiply just as sign $+$ tells you to add.

3. Make multiplication problems for examples 1 to 6. Make them about children walking by twos. (Pictures shown.)

Example 1. There are 2 children to a pair. Four pairs will be 8 children because four twos are eight.

(a) $4 \times 2 =$
(b) $3 \times 2 =$ etc.

D. *Learning the multiplication facts*

1. Learn these facts for 2's:

$$\begin{array}{r} 2 \\ \times 1 \\ \hline 2 \end{array} \quad \begin{array}{r} 2 \\ \times 2 \\ \hline 4 \end{array} \quad \begin{array}{r} 2 \\ \times 3 \\ \hline 6 \end{array} \quad \begin{array}{r} 2 \\ \times 4 \\ \hline 8 \end{array} \quad \text{etc.}$$

2. The answers to addition examples are called sums. Answers to subtraction examples are called _____.

   The answers to multiplication examples are called products. You need to know products just as you needed to know sums. Study the multiplication facts many times.

E. *Using multiplication facts to solve problems*

1. A list of problems using multiplication facts for 2.
2. Make six multiplication problems using pairs of boots.

F. *Developing the equality of product resulting from interchange of multiplier and multiplicand* ($4 \times 2 = 2 \times 4$)

1. A picture showing four rows of toy soldiers with two soldiers in each row furnishes the setting for:

   "Bill's desk has _____ rows of soldiers with _____ soldiers in each row. That is _____ soldiers in all."

   $4 \times 2 =$ _____. (Four 2's = _____)

   "Bill is looking at his soldiers from the side of the desk. He does not see four rows but _____ rows with _____ soldiers in each row."

   $2 \times 4 =$ _____. (Two 4's = _____)

   If you know $4 \times 2 = 8$, you know $2 \times 4 = 8$.

2. This picture $\begin{array}{c} x\,x\,x\,x\,x \\ x\,x\,x\,x\,x \end{array}$ is for 2×5 and 5×2.

   Make pictures like this for 4×2, 2×4, 2×3, 3×2, etc.

3. Nearly all multiplication facts go in pairs. If you know one fact of a pair you know the other.

G. *Developing multiplication facts by adding*

1. The equality in such situations as 2×4 and $4 + 4$ is demonstrated.
2. In these examples find the product by adding:

   Two 4's, two 8's, etc.

H. *Learning more multiplication facts*

Learn these facts.

$$\begin{array}{r}1\\ \times 2\\ \hline 2\end{array} \quad \begin{array}{r}2\\ \times 2\\ \hline 4\end{array} \quad \begin{array}{r}3\\ \times 2\\ \hline 6\end{array} \quad \begin{array}{r}4\\ \times 2\\ \hline 8\end{array} \quad \text{etc.}$$

I. *Exercises for study and practice*

1. A list of situations to be solved without pencil and paper. The following is typical. "In each of the 4 boats on the lake there were 2 people. There were ----- people in all the boats."
2. $? \times 2 = 10$, $4 \times 2 = ?$ $2 \times ? = 6$, etc.
3. Make practice cards for facts with 2. Place cards for facts you do not know perfectly in an envelope. Look at them often. Take cards out of the envelope when you know the facts. See how quickly you can learn all.
4. Write the answers.

$$\begin{array}{r}4\\ \times 2\\ \hline\end{array} \quad \begin{array}{r}2\\ \times 6\\ \hline\end{array} \quad \text{etc.}$$

Outline of Sample Program III

A. *Introduction to multiplication situations*

1. Answering questions in multiplication problems by indirect methods: Write the answers to the questions in the six problems in this exercise. If you do not know the answer, make a drawing to show what the problem tells with words. Then, count if you need

to. Wait until the first two problems are read before you start.

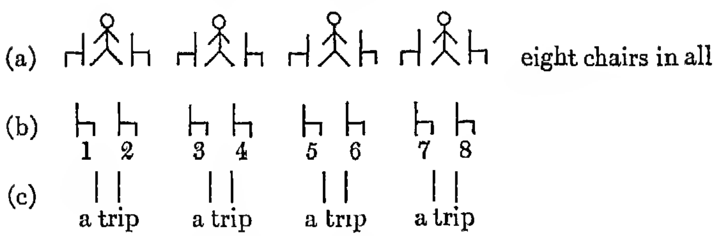
Problem 1. On each trip to the stage John carried two chairs. How many chairs did he carry in four trips?

2. In the remaining five problems other multiplication situations are used.

B. Diagram and number records to show that the answers to problems are correct

1. With marks or a drawing show that your answers to questions in the problems are correct. Be sure your drawing shows the same thing that the problem tells with words, and that it shows how you thought in getting your answer.
2. Show with numbers what your drawing shows.
3. The next step is a class evaluation of the drawing solutions or diagrams and the number records. The chief criteria in the evaluation are, "Does this show the same thing as the problem told?" "Does it answer the question?" "Can you tell from this record how the person thought in getting the answer?"

Three typical diagrams are shown here



Two typical number solutions are *a* and *b* below:

(*a*) $2 + 2 + 2 + 2 = 8$ (*b*) $\begin{array}{r} 2 \\ 2 \\ 2 \\ \underline{2} \\ 8 \end{array}$

C. *Developing statements of multiplication facts from diagram and number solutions to problems*

1. Six problems involving additional multiplication facts — procedure to be used similar to that used in A and B above.
2. Each of your number records and diagrams for the problems you have worked shows a number fact. Problem 1 shows that four 2's are 8. Look at your record to see if you show that "four 2's are 8." Write the number facts you have shown in your records for the other problems.
3. Number facts such as "four 2's are 8" are called multiplication facts.

D. *Practice exercises*

1. Give the answer to each of these questions (to be answered orally): "How many are five 2's?" "How many are four 3's?" etc.
2. Write the multiplication fact for each of the oral questions above.

E. *Writing and reading multiplication facts*

1. The sign for multiplication is "×" read "times." "Four 2's are 8" can be written $4 \times 2 = 8$ or $\begin{array}{r} 2 \\ \times 4 \\ \hline 8 \end{array}$

   You may read this, "Four twos are eight," "Four twos equal eight," or "Four times two equals eight."
2. Practice reading these facts.

   $4 \times 2 = 8$ $\qquad \begin{array}{r} 2 \\ \times 5 \\ \hline 10 \end{array} \qquad \begin{array}{r} 3 \\ \times 4 \\ \hline 12 \end{array}$ etc.

   $5 \times 3 = 15$
3. Write these facts, using only figures and signs. Four twos are eight, three threes are nine, etc.
4. Write all the multiplication facts in your list, using only numbers and signs.
5. Cover the last number in your multiplication facts. Then see if you can give the whole fact.

F. Finding the best way of answering multiplication questions

1. Write the answer to the question in each of these problems:

   Problem 1. The teacher gave 3 sheets of paper to each of 4 boys. How many sheets of paper did she give to all? (Three other problems are provided.)

2. Show three ways of getting the answer to the questions in the four problems above.
3. In these ways of getting the answer you have used diagrams, addition, and multiplication Which is the best?

   Because of its brevity the multiplication method was selected as the best.

G. Development of more multiplication facts

1. Write answers to questions in these problems. Then show with numbers what number fact you used.
2. In answering the questions in the problems one boy wrote the number question first for each problem. Here are his questions·

   Problem 1 $7 \times 3 = ?$ etc. Did he write the correct number question for each problem?

3. Write the number questions for the problems below.

H. Seeing how well you can read multiplication facts

1. Read the facts in each column: First column. $2 \times 2 = 4$, $3 \times 2 = 6$, $4 \times 2 = 8$, etc. Second column: $2 \times 3 = 6$, $3 \times 3 = 9$, etc.
2. Begin at the top of column 1 and read the last number of each fact. When you do that you are really counting by 2's
3. To solve problems quickly you need to know the facts. Cover the last number in each fact and then see if you can give the whole fact.
4. Turn your paper over and write as many of the multiplication facts as you can remember

I. Practice exercises

1. In this oral exercise answer each question as quickly as you can:

   "How many are eight twos?" etc.

2. Write the answer to each of these number questions:

   1. $2 \times 2 =$ 2. $6 \times 3 =$ etc.

3. Show with marks that your answers to questions 2, 5, and 9 are correct.

J. A timed exercise

1. Write the answer to each of these number questions as quickly as you can. When you finish raise your hand. (All multiplication facts previously studied are included. The teacher keeps time.)

2. A class discussion in which the main topic is the different ways of answering used by the pupils. Those who get answers quickly will usually say they just knew, while those who work slowly will tell how they used some roundabout way.

K. How to study multiplication facts

1. The timed lesson showed you that knowing the facts saves time. Besides, to do your arithmetic well you need to know the facts. Do you have a good way to study facts? Remembering how you studied addition and subtraction facts may help you to find a good way to study multiplication facts. Write three ways of study that might be used in learning such new multiplication facts as

$$\begin{array}{r} 4 \\ \times 8 \\ \hline 32 \end{array} \quad \text{and} \quad \begin{array}{r} 5 \\ \times 9 \\ \hline 45 \end{array}$$

2. A listing and discussion of ways of study

   From class list and that of other classes the following list is prepared:

   (*a*) Write as many facts as you can think of.

   (*b*) Make a fact table putting all 2's together, 3's together, etc.

(c) Study the facts by saying them over and over.
(d) Cover the answer to each number question and then try to give the answer. Then, look, read, close eyes, and say.
(e) Study flash cards of multiplication the way addition facts were studied.[1]
(f) Touch the desk as you give answers.[1]

3. How would you use way *c* in studying $8 \times 3 = 24$? Tell exactly what you would do.
4. How would you use way *e* to study $9 \times 4 = 36$?

L. Memorizing the facts

1. With pencil and paper do study suggestion *a*. (See above.)
2. With pencil and paper do study suggestion *b*.
3. Use study suggestion *c* with the facts you wrote in 1.
4. Use study suggestion *d* with the facts you wrote in 1.
5. Use study suggestion *f* with the facts you have written for *a*.
6. With the cards use study suggestion *e*.

M. Reason for and exercises in overlearning

1. Unless you do something to stop it, you will forget some of the facts you have just learned. One way to keep from forgetting is to study the facts again and again after you already know them. Suppose you already know that "six threes are eighteen" Studying "six threes are eighteen" by saying it will keep you from forgetting. Study all these facts again:

$$\begin{array}{r} 2 \\ \times 4 \\ \hline 8 \end{array} \qquad \begin{array}{r} 3 \\ \times 6 \\ \hline 18 \end{array} \qquad \begin{array}{r} 2 \\ \times 2 \\ \hline 4 \end{array} \quad \text{etc.}$$

N. Using multiplication facts in the solution of problems

1. You will use some of the things you learned in answering the questions in these problems.

[1] For a complete description of the study methods *e* and *f*, see Chapter 5.

(A list of eight problems involving multiplication facts.)

2. Prove your answers to problems 2 and 6.

An Analysis of the Sample Learning Procedures

In the three sample procedures outlined in the preceding section some marked likenesses and differences appear. The chief likenesses are the use of questions about quantitative situations as a means of introducing multiplication, the use of examples for practice, the explanation of signs and how to read multiplication facts. The differences among the three samples are much more varied than are the likenesses. Because some of these variations illustrate disagreement on fundamental methods, a discussion of important differences is provided. In this consideration of differences the reader should keep in mind that sample procedures I and II were taken directly from textbooks and are therefore very brief, while Sample III was not subject to the limitations of textbook presentation.

1. The most important difference among the three samples is in the experiences provided for the child in the introduction to each new phase of the process of multiplication. In Sample I the child becomes acquainted with each new step primarily by having it worked out for him or by having the procedure to be used prescribed (see Sample I, *A*, *B*, *C*, and *E*). Likewise, in Sample II the child becomes acquainted with new steps primarily by having them worked out for him, but there are also provided some exercises for the child to do himself (see II, *B* 3 and 4, *C* 3 and *F* 2). In Sample III the child has an opportunity to work out a new thing before it is explained to him (see III *A*, *B*, and *C*). The first two samples, then, stress demonstration and telling in the teaching of facts, whereas the third program emphasizes the use of problem situations to create a need for the facts to be taught and to provide a background of experiences for the demonstration and telling that is used.

2. The emphasis given to things for the learner to do is another important difference in the sample procedures. Sample I provides the least active participation by the learner, section *C* 3 being the first specific suggestion for the learner to do much more than to listen to or follow the explanations provided. For other suggested things for the learner to do see Sample I, *C* 5, *D* 2, 3, *E*, and *G*. Sample II directs the child to do many things (see *A* 3, *B* 3, *C* 3, *E*, *F* 2, *G* 2, *H*, *I* 1, 2, 3, 4). Sample III, as has already been pointed out in 1 above, suggests things for the learner to do as the first step in the teaching of multiplication Throughout the program there is primary emphasis on having the learner do things (see III *B*, 1, 2; *C* 1, 2; *D* 1, 2; *E* 2, 3, 4, 5, *F* 1, 2, 3; *G* 1, *H* 1, 3, 4; *I* 1, 2, 3; *J* 1; *K* 1, 3, 4; *L*, *M*, *N*).

3. The initial use of problems in the first two samples differs greatly from the use of problems in the third sample. In the third sample, problems are used to create a situation in which the learner may develop facts, to provide experiences which will enable the learner to see a need for facts, and to provide experiences that will prepare the learner for explanations and demonstrations of facts In the first two samples, problems are used to provide a setting for the instructor's development and explanation of facts.

4. Different methods are employed in the three samples to help pupils see the relation between multiplication and other known processes. Sample I demonstrates the relation between adding and multiplying (see I, *B* 1) and provides for the use of counting through the provision of pictures, but the counting method is not suggested directly. The learner is never directed to use either adding or counting in the solution of problems and examples. Sample II demonstrates the use of adding and counting and suggests that the learner use these processes in the solution of examples and problems. Sample III not only suggests the use of counting and adding, but makes the solution

of problems by these processes the foundation experience for the development of multiplication facts (see III, *B* and *C*).

5. Through a timed exercise, Sample III provides the learner with an experience which shows the need for knowing facts. Sample II tells the pupil to learn the facts, as he will need to know the multiplication facts just as he needed to know those of addition. Sample I just tells the pupil to learn the facts.

6. The three samples differ widely in the methods of study suggested in fixing for automatic mastery the multiplication facts. In the first sample nothing more than "learn" is given. A footnote to teachers suggests that cards can be used after a time as a means of oral drill. In the second sample "learn" is also given without amplification, but later study of cards (see II, *I* 3) is suggested. In Sample III methods of study are made the topic of a lesson and from these and from the children's suggestions six ways of study are provided (see III, *K* 2).

7. The third sample differs also from the first two samples in the use made of proof (see III, *B* 1 and *I* 3).

While there are other differences, the seven listed above represent the important ones. It can be seen that the first two samples emphasize demonstration and explanation as a means of developing the facts for the learner. The third sample emphasizes the provision of experiences through which the learner has an opportunity to develop the facts for himself. Because of the emphasis on demonstration in the first two samples, that type of teaching will be called the *demonstration method* of teaching. Because reliance is placed on experiences in the third sample, that type of teaching will be called the *experience method* of teaching. It should be noted that while the terms *demonstration* and *experience* partially describe each method, these names are used primarily for convenience in referring to the methods in this discussion.

The differences between the demonstration method and the

experience method of teaching become significant when the two methods are evaluated in terms of the principles governing the selection of learning experiences discussed earlier in this chapter. For such an evaluation the following questions are suggested.

Which method gives the pupil the more experiences for learning? As was shown in difference number 2, the experience method gives the more opportunities to learn by doing things.

Which method gives the pupil more opportunities to see sense in what he does? Here again the answer is the experience method. In the creation of needs for learning by use of problems, cumbersome methods of writing, timed tests, and the like, the experience method gives better opportunities than the demonstration method for the child to see sense in what he does.

Which method gives a better opportunity to see the relation between what he knows and the new facts? A look at difference number 4 will show that the experience method gives the better opportunity.

Which method gives the pupil better experiences as a basis for the formulation of generalizations? The experience method, with its emphasis on various ways of solving, on proof, and on giving the pupil experiences which will create a need for the fact or process being taught, provides better background for the formulation of generalizations than does the demonstration method.

Before proceeding to further consideration of the experience and demonstration methods, it should be recalled that the samples which illustrated the demonstration method were taken from children's texts. Such texts, because of the limitations of space and because many teachers wish to use their own methods, do not provide detailed teaching methods. If, as has just been indicated, some teachers do not use the method of the text, the superior features of the experience method may be practiced even where texts based on other methods are used.

The sample procedures were necessarily brief and dealt with only one phase of the teaching of arithmetic. Therefore, not all the major differences between the two methods of teaching were illustrated. In the next section other characteristics of the experience and demonstration methods of teaching are contrasted.

Demonstration and Experience Methods Contrasted

The demonstration method and the experience method of instruction, as already illustrated in the teaching of multiplication, show that while both methods introduce the teaching of a new phase of arithmetic by means of a problem, they differ widely in the use made of that problem in the instructional process. In the experience method the problem is a means of getting the learner to see the relationship between the new phase of arithmetic and the arithmetic he has already mastered. This is accomplished by permitting the learner to use processes with which he is familiar (in the sample, adding and counting) to solve the problem and then directing him to work for a better method of solution. In case the best method is not discovered by the learner, he is shown this best method, but he is then directed to prove that it is another way of getting the same answer which he got by the solution he already understands.

In the demonstration method, problems are primarily a means of providing a setting for the demonstration of the new phase of arithmetic that is being taught. The quantities involved in the problem are pictured, and, with the aid of these pictures, the new process is carefully explained for the learner by the textbook or by the teacher. The text or teacher may also develop for the learner the relationship between this new phase of arithmetic and the arithmetic already studied.

Careful study of the uses of problems as contrasted in the foregoing explanation will show that the experience method

makes better provision for individual differences among learners than does the demonstration method While the experience method permits the learner to start the solution of the problem by utilizing what he already knows, the demonstration method requires all learners to employ the same procedure.

The experience method, with its emphasis on provision of opportunities for the pupil to figure out answers for himself, develops independence on the part of the learner. On the other hand, the demonstration method by its careful explanation makes for dependence of the learner on the textbook or the teacher. The experience method, by emphasizing, in the initial work on a new phase of instruction, processes already mastered, develops in the learner a feeling of confidence. This emphasis on processes already mastered, and the development by the learner of the relationship between the old process and the new, not only make for understanding, but they are also a means of showing the learner that arithmetic is a closely knit system.

In the demonstration method, imitation of the text or teacher is a major factor in learning. In the solution of problems, the demonstration method puts much reliance on working by analogy, and thus puts a premium on the learner's ability to follow the thinking of others. In the experience method, the use of diagrams, processes, and study procedures already mastered plays a major role in learning. While the learner is offered methods of study for imitation, he is not required to use only these means and exclude all others. In the solution of problems the experience method puts reliance on diagrams or pictures and the use of known even though cumbersome computational procedures, and thus puts a premium on pupil initiative.

In the demonstration method little attempt is made to acquaint the learner with the specific purpose of various exercises. In the experience method, a definite attempt is made to show the learner the purpose of the various exercises he is directed to do. A reason for doing them is provided. For example, the timed

test shows that those who know the facts can do work more quickly.

Further discussion of the basic differences between the two methods of instruction is not essential to the purposes of this book. Sufficient evidence has been presented to show why the writer believes the experience method is superior to the demonstration method.

Before proceeding to other phases of arithmetical instruction, the student of the teaching of arithmetic should consider the statements of authorities concerning methods of teaching. A list of selected references will be found at the end of this chapter. Special attention is called to the *Sixteenth Yearbook* of the National Council of Teachers of Mathematics. In the chapter on learning in this *Yearbook*, McConnell[1] states:

> Meaningful learning emphasizes discovery and problem-solving. In fact, from this point of view, learning is thinking. Instead of learning "facts" and then using them in thinking, we can learn "facts" by thinking. This doctrine means that learning should be characterized by insight, and it sharply condemns the traditional practice in arithmetic of having children memorize certain operations in abstract form, in order to apply them in verbal problems afterwards . . .
>
> Instead of authoritatively identifying correct responses for children, courageous teachers are now encouraging active exploration and discovery and self-directed learning.

A thought-provoking statement on methods of teaching mathematics is found in Westaway's *Scientific Method*:[2]

> Let us suppose that a teacher, finding his class unable to solve a particular mathematical problem he has set them, "works

[1] T. R. McConnell, in *Sixteenth Yearbook*, National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1941), p. 284. Other statements concerning the role of discovery and experience in learning arithmetic will be found on pages 71–79, 107–18, and 268–89 of this *Yearbook*.

[2] F. W. Westaway, *Scientific Method* (London: Blackie and Son, Ltd., 1931), p. 460.

out" the problem himself. No doubt the pupils will, as a rule, be able to follow without difficulty the different steps of the solution and be convinced of its accuracy But what have they gained? They have had no real share in the work. They are still ignorant as to the way in which the teacher discovered how to solve the problem. To work through, in this way, a whole problem for a class is not only unnecessary; it is a teaching blunder of a serious kind. All that the pupils really require, or, at all events, ought to be given, is a clue by which they may set to work themselves. The art of teaching mathematics consists, at bottom, in telling the children just enough, but no more, to enable them to initiate a plan for successfully assailing the central difficulty of a problem. To tell them more than this is, psychologically, altogether wrong.

Although, when a problem has to be solved, the procedure to be followed is that of analysis, it is a fallacy to think that, for effecting such an analysis, explicit and final directions can be given which would enable a pupil to proceed, with certainty of success, to the solution of any proposed problem or to the demonstration of any proposed theorem No such instructions are possible, though, according to the ordinary textbooks, all we have to do is to assume the truth of the theorem or the solution of the problem, deduce consequences from this assumption, and combine them with results which have already been established. If a consequence can be deduced which coincides with some result already established, it may happen that by starting from the consequences which we deduced and retracing our steps, we can succeed in giving a synthetical demonstration of the theorem or solution of the problem. But such a rule is altogether too vague for general application, because no exact instructions can be formulated by which we are to combine our assumptions with results already established.

The reader will find that some aspects of the method of teaching outlined in this book are similar to the heuristic method described by Young[1] Examination of Young's discussion of methods, especially the synthetic and analytic, should result in

[1] J. W. A Young, *The Teaching of Mathematics in the Elementary and the Secondary School* (New York: Longmans, Green and Company, 1907), chap IV, pp 69–80, see also Westaway, *op cit*, chap XLVI.

a better understanding of the relative merits of the two methods of teaching presented in this chapter. The method of instruction described in this book is, of course, an example of the inductive method. In considering its merits, then, the reader might profit from a study of inductive and deductive methods of teaching.[1]

In his well-known books on the teaching of arithmetic, Morton[2] states: ". . . new steps and processes should be discovered by the children from their relationship to steps and processes already learned." Laisant, the French writer, characterizes teaching by stating that the goal of school instruction is ". . . to interest the pupil, to induce research, to continually give him the notion that he is discovering for himself that which is taught him."[3]

The method of teaching proposed in this book is closely related to the problem attitude of learning as outlined by Dewey. According to Dewey it is better for pupils to do their school work (learning) as seekers, conscious of problems whose solution satisfies some need of their own, than it is for pupils to do work (learning) in which they do not see any value. In the case of arithmetic, this need to be satisfied is first a solution of the problem and then a discovery of the best solution. In this way, children are given a purpose for learning. As Thorndike describes the procedure, "The important requirement is that the pupils should be aware of the problem and treat the manipulation as a solution of it, not as a form of educational ceremonial which they learn to satisfy the whims of parents and teachers."[4]

[1] L. B. Earhart, *Types of Teaching* (Boston: Houghton Mifflin Company, 1915), chap. V, or Paul Klapper, *The Teaching of Arithmetic* (New York. D Appleton-Century Company, 1934), chap. XV

[2] R L Morton, *Teaching Arithmetic in the Elementary School* (New York· Silver-Burdett Company, 1937–38), II, 13.

[3] M Laisant, *La Mathématique* (Paris Carre et Naud), pp. 188–89.

[4] E. L Thorndike, *The Psychology of Arithmetic* (New York: The Macmillan Company, 1922), p. 268.

In Dewey's analysis of the complete act of thought, the first step is the realization of a need or a difficulty to overcome. As Horn has repeatedly pointed out, needs or difficulties do not arise or grow out of thin air, they arise out of a situation. The big task of instruction is the placing of children in situations out of which a difficulty will arise The method of teaching proposed in this book makes extensive use of problem-setting to provide such situations. Obviously, the difficulty confronting the learner is not merely the solution of the problem; it is also the finding of the most easily understood, the most economical, eventually the best solution. To amplify and supplement these sketchy statements from the writing of authorities, the reader should consult the references cited.

Experience Method, Discovery Method, or Inductive Method

In the discussion of statements of authorities concerning method, it should be noted that no direct reference was made to either the experience method or the demonstration method. Special attention is called to this omission because the use of a name, such as *experience*, to identify a method of instruction is subject to important limitations. The reader will recall that, when the terms *experience* and *demonstration* were first used, the statement was made that these terms were being introduced primarily for convenience. The name *demonstration method* was assigned to the most common methods of instruction now in use in textbooks, and that of *experience method* was assigned to the mode of instruction advocated by the author of this book. In general discussions where reference is frequently made to two differing methods, epithets such as *experience* and *demonstration* are quite helpful. Continued use of the terms is, however, a questionable procedure. Since each partially describes only one aspect of the method it is used to identify, the uncritical

reader is likely to assign that one characteristic exclusively to the method bearing the name. For example, he might infer that demonstration is completely excluded from the experience method. "All or nothing" statements of that type are detrimental to teaching. The experience method, as described in preceding sections, does involve some demonstration, and certainly the demonstration method as described involves experiences. The term *experience method* should, then, be used with care, if at all.

The inadequacy of a single term like *experience method* can be seen more easily if the aptness of other names for the same method of instruction is considered. The mode of instruction advocated in this book certainly gives to discovery a significant role. It might, therefore, be called the *discovery method* — a term which in fact is most commonly used by those who follow this method of teaching. The method does not, however, depend on discovery as the only means by which a child is to learn a new process. The teaching procedure advocated in this book is inductive in its approach and might therefore be called the *inductive method.* The fact that three names can be properly assigned to one method of teaching is sufficient evidence to make us suspect that methods of teaching are too broad to permit the use of a single term to describe any of them.

Fortunately, we need concern ourselves no further with one-word identification of different methods. As far as methods are concerned, only one is advocated here. The need for reference to another method in this chapter is self-evident. Hereafter, however, when the term *method* is used in this volume, it will refer to the method of teaching advocated by the author.

The Use of Proof in Instruction

The principles presented in preceding sections were sufficient to permit an evaluation of the teaching procedures described. However, the procedures described do not represent the entire

field of arithmetical instruction. The remaining sections of this chapter present further principles and characteristics of method.

In the description of recommended teaching procedures (page 35), one step required the children to prove or demonstrate their understanding of a fact. This proof usually consists of the actual regrouping of objects or marks to show the fact. For example, the fact that $6 \times 4 = 24$ is shown by taking six groups of four objects each and rearranging them into two groups of ten each and one of four. Pictures and diagrams and the use of measurement are also frequently used as a part of such proof. From these examples it can be seen that proof is dependent upon the senses of perception. Obviously, this meaning of the word *proof* is not the same as that used in higher mathematics where the truth of a statement is established by relating one idea to another. The term *proof* as used here has the general meaning of furnishing evidence, data that are understood by the persons concerned, which will aid in reaching a decision about a doubtful issue.[1] Children understand the use of the terms *prove* and *proof* as they are used in arithmetic, therefore those terms, rather than the terms *demonstrate* and *demonstration*, will be used in this discussion. Since the type of proof suggested is dependent upon the senses of perception, most proofs should be made with relatively small numbers.

Requiring proof will force the child to use the longer, more concrete procedures, which, in beginning arithmetic, are very important to understanding. Later in this book, where actual teaching procedures are described, frequent use will be made of proof as a means of checking the child's understanding. (See lesson 2, Chapter 14.)

Requiring proof is also a good method for getting the child to see his own errors. Consider, for example, the case of the child

[1] J. W. A Young, *The Teaching of Mathematics in the Elementary and the Secondary School*, pp. 109–10, 125–27, 229.

who says that the sum of 7 and 5 is 11. If this child is told to take one group of 7 objects, put with it another group of 5 objects, and find the total, it will not be necessary for the teacher to tell him that his statement was wrong. The discovery of errors in this way makes a good learning situation.[1]

Many textbooks suggest checking of various kinds as a means of proving. For example, the correctness of an addition of two numbers is checked by subtracting one of the numbers from the sum. The resulting remainder should be the other addend. In a similar manner addition is used as a check of subtraction, division as a check of multiplication, and multiplication as a check of division. Another means of checking addition is to add in the opposite direction; in multiplication the interchange of multiplier and multiplicand is sometimes used as a check. But such forms of checking do not provide a proof that is easily understood by children. In fact, proof of this type is usually more difficult for them to understand than the original solution. Checking, then, does not provide good evidence of understanding, which is the major purpose of proof.

The Use of Oral Arithmetic

According to the purposes listed earlier in this chapter, the contribution of arithmetic to thinking is arithmetic's major value in education. Much of this thinking is not the type that is done with pencil and paper in hand. It is primarily the kind that is done when numbers are read or heard with understanding, when the approximate ratio between quantities is secured, when quick judgments involving quantity are made. This brand of thinking must function without the aid of a written record of thought. Fortunately, most of the computations of everyday life involve calculations that do not require the use of pencil and paper. In order to be as nearly lifelike as possible,

[1] Examples of children's proof are found on pages 156, 174, 175, and 196–97.

therefore, the learning exercises in school arithmetic should make extensive use of oral or unwritten calculations.

In addition to being more nearly in line with practices outside the school, the use of oral arithmetic helps to overcome one of the most serious criticisms of arithmetic instruction — the charge that the school program hurries children into the use of numerals. This hurrying has led to mere verbalistic learning of symbols. For example, children just beginning the study of arithmetic have been known to read $4 + 3$ as *seven*, just as they read the symbols *m o t h e r* as the word *mother*. In such instances children do not recognize that the numeral 4 stands for four objects, or that the statement means a group of four and a group of three are made into a single group containing seven. Number symbols used verbalistically in the beginning of the study of arithmetic are a block to understanding; oral arithmetic helps to guard against the development of such meaningless learning.

Unwritten arithmetic has an additional advantage of emphasizing the use of number relationships and other important aspects of the number system. Unless unseen numbers are very small, our thinking about them deals with approximations of values and with estimates of results. Of necessity, in making mental computations we compare less familiar numbers, such as 49, 97, 504, and 986, with numbers which are well-known benchmarks, such as 50, 100, 500, and 1000, in our number system, and we make our calculations with the round numbers because they are simpler to handle than others. Suppose, for example, that we need to know the sum of two quantities represented by 47 and 46. One of the first generalizations we would make is that the sum of these two is a little less than 100, since each of the numbers is a little less than half of 100, or 50. If we wished to know the exact sum, we should probably compute it most readily by adding the tens of one number to the tens and ones of the other, and then adding to that sum the ones of the first

number, or we might rearrange the numbers by taking enough ones from one of the numbers to make an even number of tens in the other, and then add the two new numbers thus obtained. In using the latter method of finding the exact sum, we should be applying the principle of the carrying process in addition. In many situations, exact answers are not needed. In all such cases, it is with the more significant parts of numbers — the tens, the hundreds, the thousands — that we are concerned.

Learning exercises, then, must be provided to give the child an opportunity to use numbers orally. Situations that illustrate the value of approximation must also be used. Although oral arithmetic is probably most important at the primary-grade levels, it is important enough to be made part of the learning exercises in all grades of the elementary school.

Another important function served by oral arithmetic is its use in keeping a class a unit, in the preservation of class spirit. For example, consider the situation in a classroom thirty minutes after a general assignment has been made. Because of the different rates of speed at which children work, the members of the class will have arrived at many different places in the assignment. Suppose that, at the end of the thirty minutes, the teacher uses an oral exercise. All pupils become again one class giving attention to the same thing. When used in this way, oral arithmetic is a unifying experience. Oral arithmetic provides an easy means of getting the attention of pupils. It requires no pages to be located, no words to read, and there is no need for the teacher to take his eyes from the class. The oral arithmetic period also provides a rapid means of reviewing and practicing important arithmetical facts and processes.

To be successful in oral arithmetic, pupils have to give strict attention and learn to select the crucial points. Learning to give attention and to select the important points are valuable traits which arithmetic instruction can afford to cultivate.

As will be shown in later chapters, oral or unwritten arith-

metic may also be valuable in the introduction of new processes and in problem-solving. It gives the teacher an opportunity to emphasize important aspects of number not ordinarily included in a textbook. For example, the use of a standard for reference (see pages 244 ff.) and of simple ratio (as in comparison of two cities having populations of 151,085 and 98,324 people respectively) can probably be emphasized adequately through oral arithmetic. Oral work also makes it easy for the teacher to incorporate some recreational and historical arithmetic into the program.

Use of Pencil and Paper

Pencil and paper or chalk and blackboard are generally assumed to be indispensable to the teaching of arithmetic. While these materials are a distinct aid to an instructional program and will be recommended continually, some disadvantages may result from their use. In the section on oral arithmetic it was pointed out that early use of numerals in written form, such as 4 + 3, often leads to a verbalistic learning, brought about by requiring children to work with written numerals in learning number facts before they have had enough experience with the concrete (actual objects) and semi-concrete (marks or dots) representation of the quantities for which the numerals are symbols. Furthermore, the extensive use of pencil and paper tends to overemphasize the exact kind of arithmetic which demands the correct answer to a problem. Because most of our common uses of number occur in situations where exact computations are not needed, such emphasis is not justified For example, we don't have to do an exact calculation in order to decide whether the thirty-five miles between us and the next town is too great a distance to cover before lunch, nor do we need to set down the exact figures and perform a long-division problem in order to see that the attendance of 34,381 customers at the Bear-Packer game is about three times the number, 12,109, that attended the

Ram-Cardinal game. In making such judgments, we substitute "round" numbers for exact numbers to simplify the job of computing. We approximate the given values and get along without writing down any figures.

Probably pencil-and-paper methods are overemphasized because the work done with pencil and paper can be checked later. Written records are used very much for purposes of evaluation This procedure has led to practices which are not always directed toward the best interests of the learner. While there is nothing wrong with evaluation in arithmetic,[1] far too much of the time in many classrooms is taken for securing data that can only be used to answer such questions as "Are these pupils able to do these exercises?" More time needs to be left for the actual instruction.

With these limitations or abuses in mind, consider the merits attached to the use of pencil and paper in arithmetic — merits that far outweigh the limitations and that should therefore receive special attention. The first real use of pencil and paper in arithmetic is the keeping of records, such as the date on which work is done or the number of an exercise. A little later paper and pencil are used as a method of recording thought. For example, in an exercise a child was asked to show five. He showed the five digits of one hand and recorded them by making five marks. Another child showed two fingers on one hand and three fingers on the other. His "five" was similarly recorded by five marks. A child's method of adding eight and four is another example. He thinks: "Take two from the four and put it with the eight to make ten. Then I have one ten and two, or twelve." The thought was recorded thus:

IIIIIIIIII II .
ten two

[1] See Chapter 12, "Testing in Arithmetic."

The concept of pencil-and-paper procedures in the primary grades as chiefly a means of recording thought (Thiele [1]) will prevent much of the meaningless number manipulation for which arithmetic is so disliked. The graphic record of steps in the thought process is a distinct advantage in seeing what the child has done, and often it makes clear both to the teacher and to other children something that had not previously been understood. Pencil and paper, properly used, become aids to understanding.

In the solution of problems that require the handling of numbers of more than one digit, pencil and paper serve a real need. By preserving a record of thought, they relieve the mind of burdensome remembering. To illustrate this need, try to divide mentally 5329 by 28. As arithmetical work becomes more detailed, the need increases for some means of recording the results of steps already completed.

Then, too, the paper record provides a means of checking. The use of pencil and paper in testing or evaluating already has been mentioned. This is a legitimate and economical way of securing data and should most certainly be considered an integral part of the instructional procedure.

Paper and blackboard are also of value in demonstrating to others the truth of a supposed fact or the correctness of a proposed solution. It should be recalled that one of the major points in the teaching method illustrated in the third sample (pages 34 ff.) was the recording and evaluation of different ways of solving. The discussion would be materially handicapped if no board or paper were available for demonstration of methods.

The writing of facts and processes is a well-recognized learning procedure. Just as a child learns or becomes more certain of the spelling of a word by writing it, so does his recording of an

[1] C. L. Thiele, in *Sixteenth Yearbook*, National Council of Teachers of Mathematics, pp. 49–50, 56.

arithmetic fact aid him in learning it. There is no clear explanation of how the act of writing aids learning, but no doubt the explanation of its value is that it focuses the pupil's full attention on a fact. In any case, everyone agrees that *seeing* number facts reinforces learning that is done through *hearing* and *saying* them; and *writing* facts is still another way of making learning easier, more vivid, and more permanent.

From the uses listed, it can be seen that pencil and paper and blackboard have an important place in the instructional program. The major point for teachers to keep in mind is that the primary purpose of written work is to record thought. A written record, by relieving the child of the need to remember a particular fact or series of facts, enables him to give attention to other items. If this use or reason for written work is made clear to children, they can see in it a real need for "figuring." It should be remembered that one of the purposes of the meaning theory of instruction is to provide an arithmetic program "in which the child sees sense in what he does." Writing figures and drawing diagrams to make records of thought seem to be sensible procedures. Consequently, to make pupils conscious of this major purpose of written work should be one of the goals of instruction. Here, as in other phases of arithmetical instruction, teachers must understand the reason for doing a thing, and must be convinced that the reason is significant enough to affect teaching procedures. Otherwise classroom methods will not be improved.

Generalizations in Learning

It is now almost universally recognized in psychological theory that generalizations or rules should appear late in the instructional process; that they should be the summary or culmination rather than the first step in learning new things; and that they should usually be expressed by the learner in order that he may

show his understanding of them. In the textbooks of seventy-five or one hundred years ago, rules or generalizations were the first aspects of a procedure presented. Textbooks of today first provide experiences and then give the rule. Examination of recent third- and fourth-grade arithmetics will reveal that authors do not leave to pupils the important learning experience of formulating the rule or generalization. This last statement is not intended as a criticism of textbook writers, for in the usual classroom conditions they hardly dare leave to the teachers the task of directing the development of rules.

The role assigned to the children in the formulation of statements of facts and procedures has already been discussed as part of desirable teaching methods. The learning procedures employed should make the formulation of generalizations second in importance only to the ability to apply generalizations. The best indication that can be obtained of a child's understanding of a generalization is his statement of the generalization in his own words.

Those who contend that it is the purpose of instruction to furnish the shortest and most precise statement of facts and procedures should remember that explanatory experiences and learning exercises are provided for the express purpose of leading children to an understanding of the generalization. The learning procedures advocated in this book are based on the principle that the "need for a rule should arise from experience, and the rule should be formulated as a result of experiment and observation. As the scholar gets older he will give more and more attention to the processes of reasoning in making the formulations."[1] The operation of this principle is so much a part of the teaching methods set forth in this book that further exposition here would be only a matter of duplication.

[1] *The Demonstration School Record No. II* (Manchester, England: University of Manchester, 1913), p. 217.

Natural Development of Number Abilities

Perhaps because number is so universally used, a great many people believe that much number knowledge develops naturally without the aid of instruction. The natural development of the ability to distinguish between two objects on the basis of size alone and the ability to recognize three objects as greater than two (without counting) gives some grounds for credence to the idea of the natural development of number knowledge. Further support is adduced from the fact that the fundamental number concepts develop early, and therefore adults remember little of how they obtained these concepts. The truth of the matter is that practically all the basic concepts are learned through special instruction and do not just accompany growth.[1]

Instruction in arithmetic is materially affected by the assumptions that we make, consciously or unconsciously, concerning the amount of knowledge of the subject that the pupil develops without instruction. Obviously, we do not need to teach that part of the subject which he acquires adequately without teaching. The progress of the pupil's learning is hampered, however, if we make a false assumption regarding the natural development of some phase of number knowledge. For example, if we assume that the meaning and significance of the base of our number system develop automatically as a part of the pupil's growth, and fail to teach thoroughly that important part of arithmetic work, we make a serious error. An examination of textbooks in arithmetic might well result in the conclusion that a child's knowledge of the meaning and significance of the base 10 of our number system develops automatically as the child matures. The ignorance of the majority of adults concerning this characteristic of our number system is evidence enough that such an assumption is wrong and that we should teach most

[1] Tobias Dantzig, *Number, the Language of Science* (New York: The Macmillan Company, 1930), p. 5.

carefully the number abilities which pupils do not acquire without well-planned instruction. Only a few important abilities develop normally without teaching. Those are identified and briefly discussed in the following paragraphs.

The ability to make one entity, group, or collection from related but different entities is perhaps the outstanding ability with which nature has provided us. To demonstrate this ability to yourself, give attention to the five fingers of your right hand. Each finger differs from the others. You can think of these fingers, not only as individuals, but also as a collection of five which has definite meaning as a collection When thought of as a collection, the individual fingers lose their identities in the mind of the thinker. Without this ability to think of a group of objects as one collection, man could not have invented a number system with a collection (10) as a base, nor could he have developed the idea of cardinal number.

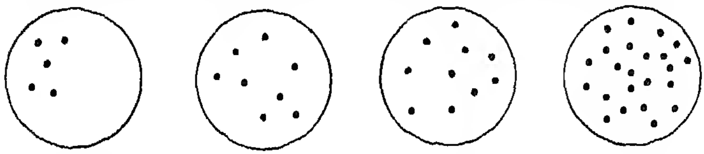
Another number ability which develops naturally is exhibited in the use of the process known as one-to-one correspondence. This is really the process of matching. Through it one can tell, for example, whether or not all children of a class are present by merely noting whether there are any vacant seats. In one-to-one correspondence other objects are substituted for or matched with the objects of the quantity under consideration. This idea or ability to substitute one object for another is very important to the efficient use of numbers. In fact, it is the foundation of all notational schemes. Counting is an advanced form of one-to-one correspondence; for in counting, a number name is substituted for each object in the quantity being counted The ability to match is important in many phases of number.[1]

The ability to determine at a glance, without counting, the exact number of objects in a small unarranged group of objects, such as two, three, four, or five, is a third of the important number abilities that develop naturally. As various studies

[1] Tobias Dantzig, *op. cit.*, p 7

have shown, children have little trouble learning to recognize instantly a group of two or of three, but they begin to experience some difficulty with four, and have much difficulty with five and six.[1] Only a few persons can give correctly and instantly the number of objects in a group that has more than six, unless the objects are arranged in a familiar pattern.

To test your ability to name a quantity without counting, try to tell instantly the number of dots in each of these circles.



Did you have to count the number in all but the circle at the left? Or compute the number by a process of regrouping? The ability to recognize the number of objects in a small group, without counting, is important in building the basic ideas of the fundamental processes and in getting the most usable concepts of the numbers below ten. Theoretically, each number which is less than the base (10) of our number system should be thought of as so many ones. Unfortunately the base of our number system is so large that it is impossible for us to see each number below 10 as a single group of ones. We are compelled to resort to some sort of grouping into smaller numbers. For example, learning to recognize at a glance that there are seven objects in a group by seeing it as a combination of four and three is much easier than by trying to see it as a single group of seven.

There are other minor number abilities which develop naturally, but those just given represent the major portion of the child's endowment from nature. Our number system is man-made; therefore, its outstanding characteristics must be learned through planned instruction. The base 10 was determined by

[1] Ned M. Russell, "Arithmetical Concepts of Children," *Journal of Educational Research*, 29: 647–63 (May, 1936)

the number of digits on a man's hands, not because 10 is in any way a superior number. Even though the base of his number system was determined by a physical characteristic of his body, man used that decimal system for hundreds and perhaps thousands of years before he invented our present convenient system of notation. After he had this system of notation, he spent several hundred years more in devising our present methods of calculating. To expect children to take the present product, to learn it without instruction, and to use it with understanding, is expecting too much. The salient features of our number system must be taught. Before teachers can present these features adequately, they themselves must learn them. In the chapters that follow, a brief treatment of various features of number and how to teach them is presented. Careful study of this brief treatment, supplemented by further reading in the references listed, will give teachers the understanding that they need for the job.

STUDY QUESTIONS

Directions. From the several responses suggested, choose the one which you consider the most acceptable answer to the question. An "N" means that none of the given choices is acceptable to you. Check your answers by referring to the text. The same procedure is to be followed with the Study Questions given at the end of each succeeding chapter.

1. Present-day arithmetical instruction gives major emphasis to which of these four objectives of arithmetic? (*1*) Role of numbers in thinking. (*2*) Role of numbers in ordering and systematizing work. (*3*) Numbers in computation. (*4*) History of number.

2. Why is the expression "two fours are eight" rather than "four times two equals eight" favored in beginning instruction? (*1*) Because we no longer use the table form. (*2*) Because pupils are not familiar with the word "equals." (*3*) Because "two fours" is the most meaningful expression. (*4*) N.

3. Which method of instruction, inductive or deductive, is used most in modern arithmetic textbooks? (*1*) Inductive. (*2*) Deductive.

4. For what purpose are problems used in children's textbooks in the introduction to a new process? (*1*) To make the work appear significant to the child. (*2*) To demonstrate to the child that he will have need for the process (*3*) To show the child how difficult the new procedure is. (*4*) To furnish a setting for the explanation of the new procedure.

5. In the six years of elementary school what subject receives more of the child's time than arithmetic? (*1*) Reading. (*2*) Spelling. (*3*) Geography. (*4*) N.

6. Numbers are said to play a significant role in thinking. Does that use of numbers necessarily imply calculation (use of such processes as addition)? (*1*) Yes. (*2*) No.

7. A child in proving that $9 + 4 = 13$ drew four marks and said 10, 11, 12, and 13. Did he prove that $9 + 4 = 13$? (*1*) Yes. (*2*) No.

8. Which of these arguments for including oral arithmetic in the elementary-school instructional program is probably the most important? (*1*) Oral arithmetic requires little in the way of instructional equipment (books, paper and pencil, and blackboard). (*2*) Oral arithmetic leads to an emphasis on the rounding of numbers. (*3*) Oral arithmetic is true to life. (*4*) Oral arithmetic makes for the best introduction to new facts and processes.

9. What major advantage does use of pencil and paper in calculation have over oral or unwritten calculation? (*1*) It is faster. (*2*) It is usually understood better by those who use it. (*3*) It tends to decrease verbalistic learning. (*4*) N.

10. If children use only a textbook, how are they to know that the methods they are taught are the best? (*1*) By accepting the author's word for it. (*2*) From careful study of all the methods suggested in the textbook. (*3*) From actual trial or test following the suggestions of the textbook. (*4*) N.

11. In the learning procedure suggested by textbooks do rules grow out of or develop from the experiences suggested? (*1*) Yes. (*2*) No.

12. Why is it misleading to apply a term such as *demonstration* to a method of teaching? (*1*) Because there is actually very little demonstration in any method. (*2*) Because the method probably has only a little more demonstration than some other methods. (*3*) Because by inference demonstration is not used in other methods. (*4*) N.

13. On what grounds is extensive use of exact computation in arithmetic criticized by some teachers? (*1*) It requires too much time to do that kind of computing. (*2*) It causes pupils to conclude that the arithmetic of school and use of numbers in life are different things. (*3*) It leads to the false conclusion that there is value in exactness. (*4*) N.

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3

Counting, Numeration, and Notation

Status of Counting

Although counting is the base upon which all later work in arithmetic rests, too little emphasis is placed upon that process in arithmetical instruction. One reason for this neglect is that many teachers have been denied opportunity to become aware of the importance of counting and of the reasons why it is important. The professional books those teachers have read and the professional courses they have taken on the teaching of arithmetic have failed to deal adequately with counting. Most of the counting exercises they have seen treat only such taken-for-granted aspects of the process as counting to find how many, thereby fostering the general notion that all there is to the learning of counting is to count. There is little wonder that these teachers have turned their attention to the teaching of other phases of arithmetic.

The process of counting is so fundamental and important to the child's understanding of and progress in arithmetic that it warrants much more careful consideration than is usually given to it. Therefore, this chapter is concerned with a discussion of the nature of counting, the importance of the process, reasons for its neglect in most schools, and suggestions for teaching of the process.

Rational Counting and Rote Counting

The term *counting* usually means one or the other of two things. To most people it means the ability to apply the number names to the individual objects in a group in order to find the total number of them. Some writers refer to this as rational counting or enumerating. To a few people counting means merely the ability to recite the number names in accepted order. Professional writers have referred to this reciting as rote counting. Unless the term *rote* or *rational* is prefixed, *counting* as used in this book refers to the use of number names in correct order to find "how many or which one."

In a recently published text it is recommended that enumeration be taught before rote counting. Brief consideration of the two kinds of counting, however, leads to the conclusion that rote counting must precede or at least accompany the learning of rational counting or enumeration Further consideration of the rote-rational counting issue will result in doubt as to whether rational counting can be taught at all before children learn to count by rote.

Consider, for example, the commonly advocated procedure of having children learn to count by enumerating objects such as blocks. In the early stages of the process, the child is taught to begin by picking up a block from a group and setting it aside as he says "one"; then picking up another block, setting it aside, and saying "two." Thus he goes on, transferring the group of blocks one by one as he says the numbers in order. At the very first, he certainly has no idea of finding out how many blocks there are in the group. He may have only a vague idea of assigning number names to the blocks He may be doing nothing more than handling a different block as he repeats, in order, the number names which he has learned to say by rote. A little later in his practice, he may be able to go through a group of objects accurately, merely by pointing to the objects one by

one as he repeats the number names in order. In this latter stage he still is not enumerating, but merely assigning a number name to each object. The truth of this last statement is often strikingly demonstrated to those who have had the experience of watching a three-year-old learning to count his fingers. Almost invariably the child will ask something like "Could this be four?" when pointing to only one finger. Such a question shows that it is a name, a means of identification, that the child wants. He is not concerned with finding how many fingers he has.

Thus the designation of objects by name, not the determination of the quantity of them, appears to be the goal and the result of the child's first attempt to use number names. To apply number names in a fixed order to a group of objects which are arranged in no particular order, the child must have command of the proper order of the names. Therefore, before the child reaches the stage of development in which he really enumerates, he makes much progress in learning to count when he learns the number names by rote through imitating what he hears his elders say. School officers and teachers should welcome the efforts of parents to teach their children to say the number names in the accepted order. They should remember that, in learning to count, three essential steps seem clearly defined: first, learning the number names in order; second, applying these names as identifying words for objects; and third, using the names in determining total quantity.[1]

Ordinal and Cardinal Numbers

The use of number names to arrange objects in order or to identify their places in a series is known as *ordinal* counting. The use of number names in serial order to find the total number

[1] Margaret Drummond, *The Psychology and Teaching of Number* (Yonkers-on-Hudson: World Book Company, 1922).

is known as *cardinal* counting. The true meaning and the place of these two forms of counting in learning and in teaching numbers, like so many other phases of arithmetic, are not always clearly understood by teachers and writers of texts. For example, one author[1] states that although both ideas of number should be taught, "the primary number meaning is the cardinal; it should be taught first." This same author illustrates ordinal and cardinal in the following manner:

O	O	O	O	(Ordinal)
1	2	3	4	
		O	OO	(Cardinal)
O	OO	OO	OO	
1	2	3	4	

Teaching the cardinal aspect of number first is not in harmony with the three essential steps children follow in learning to count. The first of these steps is to learn the order of the number names; the second is to apply these names as identifying words in arranging objects in numerical order; and the third is to use the names in determining quantity. The third step is, of course, the cardinal idea while the second step is the ordinal idea. Indeed, it would be difficult for most children to get the cardinal idea of four for the quantity OO
OO if the ordinal idea of number were not already developed. The child must know that four comes after three, or else he must know quantities without benefit of counting. While it is possible for a child to recognize groups of two, three, or four without use of ordinal numbers, the task of recognizing seven, eight, or nine would be almost impossible.

Practically all texts in arithmetic refer to ordinals only as first, second, third, fourth, and so on. Dictionaries also refer to ordinals only in that way. In ancient times people had separate

[1] R. L Morton, *Teaching Arithmetic in the Elementary School* (New York Silver-Burdett Company, 1937-38), I, 64

ordinal and cardinal names. We retain that difference today in our one, two, three, and first, second, third, but by far the greater part of our ordinal and cardinal number names are identical. To assume that ordinals are confined to the terms first, second, third, fourth, and so on, is an admission of misunderstanding of the ordinal uses of number. To demonstrate to yourself that cardinal numbers are used in the ordinal sense, consider the meaning of an index reference like "Brick, p. 246." In this case the 246 is an ordinal number. It tells the person who desires information on bricks that something about bricks is given on page 246. For that particular purpose, the position of the page in the book, not the total number of pages up to that point, is the important thing that the 246 tells. Likewise, in the ordinal sense we employ cardinal number symbols and names for such common uses as designating houses, telling time and expressing dates: 123 Pine Street, eight o'clock; June 19, 1947. We also utilize the cardinal number symbols and names in hundreds of other instances where we wish through numbers to show order, to identify, or to locate. Even in enumerating, the ordinal scheme is used to find the cardinal — the how-many-in-all. To prove this, count a given number of objects, say eleven. When you come to the ninth, which you will call nine, stop. Is this one object, which you have just pointed to or designated in some other way, nine objects? Of course not, but it is the ninth one, and therefore, nine objects in all have been identified. The last ordinal applied in identifying or enumerating a group is, therefore, the cardinal of that group. (See lesson 3, Chapter 4.)

Teachers should recognize clearly the distinction between ordinal and cardinal as indicated in the above discussion, and children should be made aware of the two uses of number even if the names ordinal and cardinal are not used. For example, even first-grade children can and should learn the meaning of page numbers. (See lesson 2, Chapter 4.)

This discussion of the terms ordinal and cardinal is really not complete without considering the origin of these two aspects of number. In ancient times man employed separate ordinal and cardinal names. Although no records of the beginning of counting are available, it seems fairly certain that the ordinal idea or plan was responsible for the beginning of counting. A reasonable theory holds that it was man's desire to identify, to order, to systematize, which led to counting. Primitive man had many uses for a method of placing things in definite order. A hunting party stalking game would find it advantageous to have the best hunters in strategic positions. Probably the one who was most skillful in throwing the spear would be first, the one most experienced in close fighting next, and so on. Similar requirements for definite order in placing men were even more important in time of war. Then, too, various religious ceremonies called for a definite order of approaching the altar, the god, or the temple. Probably the medicine men went first, then the chief, followed by the warriors arranged in some definite order. Early man's need for a universal system of identifying places is well shown by the following example of a primitive account: "On the fourth day of the journey we reached the mouth of the third of the rivers. There we waited for our scouts" Thus, in the most elementary activities of life (hunting, fighting, worshiping, and exploring) there was needed a way of identifying the position of men, objects, and events. Equally plausible uses of the cardinal idea of number do not readily occur. Even in cases where the cardinal idea would seem to be called for, as in reporting the number of warriors in a war party, the model group plan and not the counting plan of telling how many would probably have been used. Uses of an ordered system for identifying events and places, such as the examples cited above, are independent of the cardinal idea of counting. On the other hand, the cardinal idea cannot be used very effectively with quantities larger then four or five without the use of the ordinal idea. From the preceding

discussion it should be clear that the writer believes that the ordinal idea of number played the major role in the development of counting. This belief is contrary to the generally accepted idea that man learned to count in order to find out how many of certain goods he possessed. The latter theory seems unsound, since the idea of quantity in the sense of *how many* presupposes a knowledge of counting. Likewise unsound is the proposal that children want to learn to count in order to find how many.[1] The ordinal, the series idea, is the original idea and must receive the first attention.

In order to give the proper perspective to counting, it is well to consider that series rather than quantity is the more important aspect of number. As was pointed out in the section on meaning and understanding in Chapter 1, it is the ordinal or positional aspect of number that gives us our most usable concept of quantity. To test the truth of this statement, let us consider the quantity indicated by 83. Few if any of us try to visualize 83 separate units. Instead, most of us compare 83 with some well-known quantity below it, such as 50, or above it, such as 100. In other words, our idea of the quantity indicated by 83 is obtained from its position in the number series.

The cardinal idea originally developed from the custom of using model groups of familiar objects as standards of reference. To convey some idea of the quantity in his sight or in his mind, a man would say there were as many things as there were petals on a flower or pebbles in a heap. To express the idea exactly, however, he would have to match each object with a pebble or

[1] George Bruce Halstead, *On the Foundation and Technique of Arithmetic* (Open Court Publishing Company, 1912), chaps. XIV and XVIII. See also D. E Phillips, "Number and Its Application Psychologically Considered," *Pedagogical Seminary*, 5. 221–8, Tobias Dantzig, *Number, the Language of Science* (New York· The Macmillan Company, 1930), p. 8; Harry Amoss, *Rhythm Arithmetic in the Primary School* (Boston: Bruce Humphries, 1942), chap. II, H. G Wheat, *The Psychology and Teaching of Arithmetic* (Boston: D. C Heath and Company, 1937), pp. 20–23.

petal. It requires no great imagination to see the advantages and disadvantages of this scheme. The model group gave an exact measure of how many; and in the sense that a familiar, convenient, and certainly more simple group was substituted for the original, the scheme made for simplification of thought. But this scheme provided no exact means of comparing model groups of different sizes. There was no number system which the mind could use, and therefore man's idea of a quantity was dependent on unrelated model groups. All idea of size was dependent on sense perceptions.

The inadequacy of the model-group method of telling how many is illustrated by the following anecdote: Two scouts sent out in different directions by a hunting party made their report. The first said, "South of here is a herd of deer in which there are as many deer as there are feathers in my headdress." The second said, "West of here is a herd of deer in which there are as many deer as there are knots in this string." No doubt, the hunters would have been familiar with each of the two model groups used and would, therefore, have had a pretty good idea of the size of each herd. On the other hand, the hunting party could find out which scout had located the greater number of deer, or how many more were in one herd than in the other, only by matching knots with feathers. The application of the ordinal idea to the model groups permitted comparison of different-sized groups without the cumbersome matching process.[1]

Counting in the Fundamental Operations

Counting receives relatively little attention in school because school people have associated counting with poor achievement in addition and to a lesser degree with poor work in the other fundamental operations (subtraction, multiplication, and divi-

[1] For further discussion of this phase of counting the reader is referred to Wheat, *op. cit*, pp 7–12. See also Halstead, *op. cit.*, chap. III; Dantzig, *op cit.*, pp. 6–11.

sion). Children who do not know the addition combinations often resort to counting — a perfectly logical and sound method of getting the sum. Of course, such a method is a slow one, and it embarrasses the teacher by making her pupils seem stupid. However, the reason that children resort to counting is not that they have done too much counting. It is more likely that addition was undertaken before they had had sufficient experience in counting to lay the good foundation needed for intelligent mastery of the addition combinations.

The child who knows enough to count in order to find the sum of 4 and 3 possesses a great deal of understanding. He should not be denied an opportunity to use his knowledge even though the final aim of instruction is to teach him procedures that are superior to the counting process. Teachers and students of arithmetic should recognize that adding is a method of counting by groups and therefore a short method of counting. Teachers should also remember that simple steps in learning precede more complex steps. Then, instead of receiving little emphasis because poor students use it as a substitute for addition, counting should be recognized as one of the first steps in teaching addition.

Counting to find the answer is considered the foundation of all methods of solution advocated in this book. This assumption will be made especially evident when fundamental operations are studied. It is further assumed that all the problems used in introducing each of the fundamental processes can be solved by counting. The child who can count is thus assured of having at least one solution. Besides being both a starting-point and a last resort, counting is also a definite aid to understanding, primarily because it is a process in which everyone has confidence. Even adults, when working under extreme pressure, often resort to counting. When children differ about the size of a certain quantity, counting can always be used in trying to reach an agreement. The teacher confronted with

the problem of what to do when a child states that 6 and 7 are 14 will probably find few procedures as valuable as having the child find the total of 6 and 7 by counting. Thus, it can be seen why counting is considered the foundation for study of the fundamental operations of arithmetic.

Numeration

Numeration, as used here, refers to the reading of numbers expressed as numerals. (In addition to this restricted meaning, the term is frequently used in arithmetical literature to denote the process referred to previously as enumeration or rational counting.)

Soon after the child enters school, he finds that he needs to know how to read numerals in such common things as dates, house numbers, page numbers, and prices. In fact, he often meets his first reading of numerals as a regular part of the initial instruction in reading. Probably the most important single procedure used in teaching the reading of numbers is to have the pupil touch or point to numbers in a number chart (see page 94), taking each number in order and at the time of touching say its name. In teaching the reading of numbers, however, it is important to do more than develop the pupil's ability to call the symbols by their correct names. To insure his understanding of the numbers that he reads, the teaching must include fundamental procedures to develop meaning for the symbols Certainly, in developing the pupil's ability to read the number symbols from 1 to 10, the teacher should see to it that not only are the number names, three, four, five, and so on, employed, but also grouped concrete representation (actual objects) and semi-concrete representation (dots, marks, and the like) of each number. For example, in teaching the child to read for the number 3 the word *three*, the teacher might well utilize three books, three pencils, or other objects, and three marks or three dots. Or such devices as having the pupil tap three times, touch

another child three times, or say the same sound three times may prove helpful. Although the procedures just listed are primarily useful in helping the pupil to develop a concept of cardinal numbers, they are more than justified as part of the teaching of reading of numbers, for they provide repeated opportunities for the pupil to see the relation between number symbols and the ordinal values and cardinal values for which they stand.

In teaching the meaning of the larger numbers, such as 6, 7, 8, and 9, the teacher should not only employ exercises involving objects like blocks and semi-concrete representations like pictures and marks, but she should also plan the exercises so that the pupil will see each of these larger numbers as combinations of smaller groups. Such exercises should lead the pupil to see that it is easier to identify the number of objects in a large group when the objects are arranged in a combination of smaller groups. For example, after a little practice, the pupil will probably find that there is a marked variation in the amount of effort required to recognize that the quantity in each of these illustrations is 8.

(*a*) IIIIIIII (*b*) II II II II (*c*) IIII IIII (*d*) III III II

For most adults the total number of marks in *c* and *d* is easier to grasp than the total of the marks in *a* and *b*, the greater ease being due to the particular arrangement or grouping. But the teacher should remember that the arrangement of groups representing a number should not be limited to what an adult considers the best grouping. The arrangement should present a large variety of groups and children should be allowed to select the grouping that they consider best. While most adults might perhaps prefer arrangement *c*, many children would prefer *d* or *b*.

From the preceding paragraphs it can be seen that economy of thought is the major objective of the grouping of objects representing a number. The child's experience with group

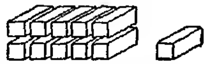
arrangements should illustrate to him the economy that results from grouping. If, for example, a child is assigned the task of giving to each of thirty classmates nine small objects, he will see the economy of counting them out by threes. The economy of grouping will be especially evident if he has to check his work. If as an additional assignment the child is required to record first by means of marks and then by means of a figure the number of objects given, he is in a position to see the advantage of a single easily-read number symbol.

As has already been implied, such exercises are concerned more with number understanding than with numeration, but there is little point in teaching children to read numbers they do not understand. Ideally, numeration should not be undertaken until children have had much experience with number ideas. Practical considerations, however, prohibit such postponement. From the day of their entrance into school, children are surrounded by written numbers. To postpone teaching the reading of numbers may deprive them of good learning situations and result in keeping useful knowledge from them.

Closely related to the reading of number symbols is the writing of numbers The two operations may be considered as parts of the same job. In learning to write number symbols, children should see the sense of having such symbols. They should have experiences which will enable them to see that the Hindu-Arabic numerals are the best means of expressing certain quantities by symbols. Such experiences can be provided by letting the child draw pictures, squares, "X's," dots, or other marks to represent quantities. (See lesson 1, Chapter 4) In order that the child may see sense in what he does, situations must be created in which the child will find a need for recording quantities. Such situations are illustrated in lessons 4, 7, 10, Chapter 4.

After the child has learned to read the first nine numbers, the collection idea of numbers discussed in Chapter 1 should be emphasized. This emphasis can be accomplished by giving the

meaning of number names, by using devices like bundles of ten sticks each, by grouping into tens a large number of objects that are to be counted, by using the tens square (page 109), and by using the abacus and the tens block.

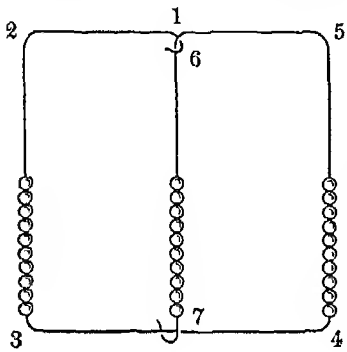


The tens block and the abacus are two devices mentioned so frequently in this book that a description of each and a brief exposition of their merits seem essential at this point. The tens block, as shown in the illustration above, is simply a small block of wood which has been only partly cut or grooved into ten smaller blocks or sticks. Tens blocks may vary in size from $\frac{3}{4}'' \times 2'' \times 2\frac{1}{2}''$ to $1\frac{3}{4}'' \times 2\frac{1}{2}'' \times 3\frac{1}{2}''$. Each of the ones blocks used with the tens block is the size of one of the ten parts in the tens block. The tens block can be made by any person who has a small bench saw. The above illustration should be sufficient to permit accurate duplication of the block. The chief advantage of using a tens block to represent ten lies in the fact that the ten ones are bound together into one collection, one entity. A glance at the illustration will show clearly how this device is of value in teaching the meaning of quantities like 11, 23, 50, etc. In addition to its use in showing an economical way of thinking of tens numbers, this device is very valuable in teaching carrying and borrowing and the multiplication and division of tens.[1]

The abacus used in the lessons described in this book was made from the wire of a coat hanger and wooden beads from the ten-cent store. Any piece of wire about 36 inches in length can be used. The steps in making it are as follows: Label one end of the wire point 1 (see diagram). Four inches from 1 at

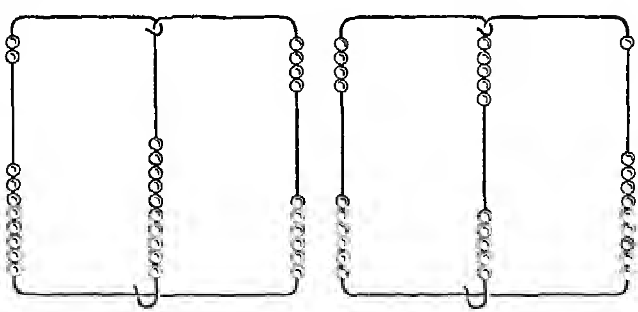
[1] For examples of the use of the tens block see Chapters 5 and 6 For a detailed discussion of the uses and merits of the tens block see Herbert F Spitzer, "A Device as an Aid in Teaching the Idea of Tens," *School Science and Mathematics*, *42*: 65–68 (January, 1942).

point 2 make a right-angle bend and put 10 beads on the wire below point 2. Next make a bend at point 3, which is about 7 inches from 2. At another 7 inches make bend 4 and then put 10 beads on the wire above point 4. Seven inches from 4, at point 5, bend again, and $3\frac{1}{2}$ inches farther make a bend at 6. Put on 10 more beads below point 6 and bend the end of the wire over the base wire at 7. Using about half an inch of end 1, bend it around the wire at 6.



The principles governing representation of quantities on this abacus are the same as those used by the ancients. Students of the teaching of arithmetic will find it worth while to consult one or more references on the use of the abacus.[1]

The following is a brief description of how the abacus can be used. The beads on the right represent ones, the middle row represents tens, and the row on the left represents hundreds.



(a) (b)

[1] Dantzig, *op cit*, chap. II, Margaret Drummond, *The Psychology and Teaching of Number* (Yonkers-on-Hudson. World Book Company, 1922), pp. 74–76, Wheat, *op. cit.*, pp. 40–47. F. H. Williams, *The Abacus and How to Operate It*, (Kelly and Walsh, Limited: 1916), pp 1–27.

To indicate quantity the beads are pushed up. The abacus in illustration *a* shows 204. The abacus in illustration *b* shows 451.

The abacus is an intermediate step between the relatively concrete representation of quantities with the tens block and the symbolic representation of quantities with numerals. Since the beads on different rows represent ones, tens, hundreds, and so on, in the same order as do the numerals in the "places" in our notational system, the abacus is an excellent device for teaching positional value.

In current practice the symbol 0 is read not only as "zero," but as "oh," "ought," "nought," and "cipher." For example, in giving the individual figures in the number 12,046 any of the above terms might be used to indicate the zero. It is difficult to justify the use of five different names for the symbol 0. Zero alone should be used except in special cases — as for example in giving telephone numbers, where custom has established "oh" as correct. In addition to the five different names used listed above the symbol 0 is of course read in still another way when it appears as an integral part of a number. For example, in reading each of these numbers, 70, 204, and 360,008, the zero affects the reading, but is not sounded as "zero" or "oh." For many years teachers have tried to teach the reading of integral numbers of more than two figures, without the use of "and." For example, they say that 106 should be read one hundred six. Outside the arithmetic classroom, however, most people read 106 as "one hundred *and* six." The latter form is more in keeping with the rules of good English and with the current emphasis on meaning. The number 10,106 is ten thousand, one hundred, and six ones. To insure clarity in speech, the plan of indicating the last of a series by preceding it with an "and" was adopted. Advocates of the omission of "and" in the reading of integral numbers wish to reserve the "and" for use in the reading of fractions — as, for example, in reading

120.6 as one hundred twenty *and* six-tenths. They say that every time that "and" is used in giving a number, the listener should know that the part following "and" is a fractional number. While this practice might have had some merit when arithmetic teaching involved dictating many long lists of figures, it no longer has much value. Even in dictation of numbers the use of "point" in decimals rather than "and" has been found to be more effective. Thus, the number 74.9 would be dictated "seventy four point nine." The practice of requiring children to listen for the "and" may even be detrimental in that it diverts attention from the significant numbers. Consider, for example, the number which you may read as "eight hundred twenty *and* two-tenths." By giving "and" the special role of indicating a fraction, the reader directs particular attention to the relatively insignificant two-tenths. The whole issue of the use of "and" is such a minor part of arithmetic teaching that no teacher should be criticized for failure to teach a child to omit the "and".[1]

The modern arithmetic text gives a great deal of attention to the reading of numbers written with figures. Major reliance in teaching is placed on the meaning of the number. The usual procedure is something like the following: "The number 380 is read three hundred eighty. The 3 means that there are three hundreds and the 8 means that there are 8 tens or 80 in the number."

After use of such initial teaching procedure as described above, the device of a chart labeling the places (see below) is often employed in teaching children how to read numbers. In such a chart the numerals in the number to be read are placed under column headings beginning at the right. The last number to the left is read first. After some experience with the chart some

[1] Dorothy Rolston and Herbert F. Spitzer, "Oral and Written Expression of Numbers of Three or More Digits," *Elementary School Journal, 17*: 116–19 (October 1941).

teachers try to get their pupils to memorize the names of the places (ones, tens, hundreds, and so on).

thousands	hundreds	tens	ones
4	3	0	8

Thus, when confronted with a number that is not easily read, the pupil can use his knowledge of the places in figuring out what the number is.

Notation

A number of references have already been made to our notational system. The Hindu-Arabic is the one we use for nearly all purposes. Briefly, this notational system consists of a plan whereby ten different symbols (numerals), nine for quantities and one to hold a place, are used to record all numbers. The importance of this system of notation cannot be overemphasized Combined with the idea of the base,[1] it comprises much of what we usually mean when we refer to our number system. For a few purposes we use another well-known system, the Roman.[2]

The superiority of the Hindu-Arabic system over the Roman system can be easily shown by comparing and contrasting the two. Both systems use a base of 10, and both use a single mark to represent the quantity one. In the Roman system a second mark is added to the first to represent two, another to represent three, and in the old Roman a fourth to represent four. Then, probably because it was difficult to distinguish between four and five marks without counting, a new symbol was used for five. The adding of ones was then repeated to make symbols for the numbers six, seven, eight, and nine. Later the scheme

[1] See pages 15–17

[2] See page 98, Selected References, *passim.* Many other systems have been used from time to time. The reader is referred to histories of mathematics for these.

was changed to include the subtraction of small amounts from large ones. IIII became IV and VIIII became IX. Every time the amount represented by known symbols became too great to be easily perceived, a new symbol was introduced. Thus, fifty came to be represented by L instead of by XXXXX, and five hundred by D instead of by CCCCC.

In the Hindu-Arabic system a separate symbol is used for each of the first nine quantities in the series. The tenth is represented by using the unit symbol again, but in a different position. This is accomplished by using a place-holder (zero) in the ones place and designating the second place as representing tens. This scheme does for notation what the base idea does for counting; that is, it goes back to the easily perceived and already known 1. In the tens place, of course, the 1 stands for a collection of ten, but is handled just as it is when it represents one unit or one. The same principle is used in expressing hundreds and other higher numbers. Thus, the value of a numeral in the Hindu-Arabic system is determined by its position. For every change of position to a higher order (one place to the left), a numeral represents a value ten times the value it had in its vacated position. For example, the numeral 2 represents successively two, ten times two, and ten times twenty as we write 2, 20, 200. No such multiplication of value is caused by moving a Roman numeral from one position to another in a number. In the Roman system the value of each numeral is always independent of its position. For example, X stands for ten, no more, no less, in XIX, LXXVII, and XLVIII. It is the principle of multiplying the value of a numeral by changing its position in a number that gives to the Hindu-Arabic system its unequaled capacity for expressing in simple form any number, no matter how large or how small it may be. The principle of position makes possible the writing of any number by using only as many different characters as the number of ones in the base of the number system. In the Hindu-Arabic system the base

is ten. We can write any number we please by using just ten characters in various positions. If we used a number system with a base of eight, we should need only eight characters. For a number system with a base of twelve, we should have to have twelve characters. In each system of this kind there must be a character for each number from 1 up to and including the number which is just one less than the base, and there must always be one character which is used as a place-holder.

For example, the number of letters in the first six words of this sentence, if expressed with numerals to base eight, would be written 34, meaning that there are three eights and four more. If written to base twelve, the number would be 24, or two twelves and four ones; and, of course, to base ten the number is 28.

The Role of Zero

The Hindu-Arabic system of notation with its positional value is made possible by the use of a place-holder, our zero. We possess no positive knowledge about the discovery of zero. We are not even certain of the people who first used it. We once thought the Arabians discovered it. Then we gave the Hindus credit for it. Now we know that the Babylonians had the idea,[1] and that they probably borrowed it from others.

Without doubt, the discovery of this principle will always rank as one of the major steps in man's climb to his present state of civilization. The idea is so simple that many teachers have never given a thought to its significance. Educators have sometimes unwittingly hidden the main point by attaching to zero connotations such as "zero facts," "zero the clown," "the tricky fellow." Some writers and research workers have even concluded that the zero facts (such as $0 + 2$, $8 + 0$, and 8×0) are the most difficult to teach. The important point to consider

[1] Edward Chiera, *They Wrote on Clay* (Chicago. University of Chicago Press, 1938), pp 154–57.

here, and to teach, is that zero is used in notation to hold a place. It is not a number in the sense that it represents a quantity. This simple fact can and should be taught as soon as children learn to write numbers. (See pages 4, 94.) The symbol 1 in 10 is no different from the 1 representing unity, except that it is in the second position from the right. When 10 is first written, the attention of children should be called to this positional difference either by direct question or through some indirect exercise. Children's understanding of 10 would be enhanced if teachers would frequently refer to it as one ten. It really is one ten while 20 is two tens. Of course, the expression *one ten* is redundant in adult language, but it helps to get across to children the collection meaning of 10.

In order to call attention to the often neglected true uses of zero, the difference between zero and the other numerals has probably been overemphasized in the preceding paragraph. It was stated that zero was not a number in the sense that it represented quantity; but in many other ways, especially in number operations, zero is a number. An excellent example of zero functioning as a number is its representation of a point on a scale such as a thermometer. The place-holder function of zero is, however, its most important use and should receive major attention. (See pages 112, 113.)

Teaching Procedures

Fortunately for teachers, most children know something about counting when they enter the first grade.[1] There is, however, much in the field of counting, numeration, and notation with

[1] Josephine MacLatchy and B. R. Buckingham, "The Number Abilities of Children When They Enter Grade One," *Twenty-ninth Yearbook*, National Society for the Study of Education (Bloomington, Illinois: Public School Publishing Company, 1930), pp 472–524, Clifford Woody, "The Arithmetical Background of Young Children," *Journal of Educational Research, 24*: 188–201 (October 1931).

which teachers can help children. Obviously, an inventory of counting ability should be taken early in the year. On the basis of the results of this inventory, different types of procedures need to be planned. With children who do not know how to count even by rote, exercises designed to stimulate attention or interest should first be used. For example, the teacher may ask the members of the class who can count to enumerate orally objects such as new books and crayons, so that the beginners may see and hear the operation. She may also have the beginners learn and repeat rhymes such as "One, two, three, four, five, I caught a hare alive," and "Here is the bee hive, where are the bees?" Of course, there is little meaning of number in the learning that results from saying such rhymes, but if the exercise motivates children to learn the order of the number names, it serves a useful purpose.

As was indicated in the section on rote and rational counting, all children learn first to count by rote. As soon as the children have learned a few number names, the teacher should encourage the use of these names by requesting different pupils in the course of ordinary school experiences to bring 2 things, to show 3 things, to make 2 things, 3 things, 4 things, and so on. Probably some reading-readiness work will require the use of number names. For example, "Color three balls blue, color two balls brown," and the like are typical of the assignments made in reading-readiness work. The ordinary wall calendar offers many possibilities for teaching this phase of arithmetic. The inferior counters may be assigned the task of putting marks on the date of each school day, then reporting daily by means of marks on the board and in other ways the number of days of school attended. Counting the number of days until a special holiday, the number of days required for seed to sprout, and the number of rainy days in a week are other examples of the type of exercise which makes use of the calendar. The possibilities of using the reading of numbers in these exercises should be

self-evident. In every first-grade room there are, of course, almost innumerable opportunities for children to do counting, as, for example, counting to find the number of chairs in the room or the number of children present. While these are good counting exercises, teachers should not assume that such exercises will furnish the initial drive or the first experience needed in learning to count. The children who can count chairs either already know counting, or they are not able to see that counting is a means of finding how many in all. Much work of the kind that arouses interest in counting should precede assignments like counting the number of days in November.

After an interest in counting has been aroused, exercises which will help to establish the order of the number names should be emphasized. Since without some crutch the human mind is able to perceive the exact number of objects in small groups only, and because we use a base of 10, the first teaching should emphasize counting only to 10. Rhymes have already been mentioned as helps in fixing the order of number names.[1] Saying the names in order is one of the best ways of establishing the order of the series. In spite of the rather meaningless procedure, children enjoy saying the number names. Since reading and other school work require the child to read numerals, exercises which employ the written numeral may be used. Placing in order a set of numbered cards is one such exercise. Putting some of the cards from a series in their approximate position, after a few have been placed, is a more advanced exercise of this type. For example, card number 1 and card number 10 are placed a suitable distance apart in the chalk tray. A child is given number 9 and asked to put it where it should go if all the cards were present. Writing the symbols in order is an exercise that may be employed with older children. Reading in correct

[1] For a list of rhymes see Rosamond Losh and Ruth Mary Weeks, *Primary Number Projects* (Boston. Houghton Mifflin Company, 1923). See also Harry Amoss, *Rhythm Arithemtic in the Primary School*, chap II

order the number names that appear on a number chart or calendar is still another exercise.

The procedure usually followed by children in learning the order of number names from 10 to 20 is probably the same as that used in learning the first 10; that is, 11 comes after 10, 12 after 11, and so on Such a procedure is not in keeping with the suggestion that teaching is to take full advantage of the number system. If that system were utilized, 11 would be taught as 10 and 1, 12 as 10 and 2, and so on. Unfortunately, the first two number names beyond 10 — eleven and twelve — give no indication of the fact that these two numbers are in reality names for ten and one and ten and two. Whether or not it would be better to approach the teaching of the numbers 11 to 19 through the ten-and-ones plan is uncertain. However that may be, if the plan of teaching is to present just the next number or the number that follows without reference to what has already been learned, an early attempt should be made to let the child see what these numbers really mean in the number system. Since some grouping is necessary for easy grasp of numbers like 11, 12, and 13. it seems logical to use the base 10 as one of the groups. Eleven, then, becomes one ten and one one; two things to be perceived instead of eleven. If this plan is to be used and accepted by children, extensive use will have to be made of such devices as the tens block. Regardless of whether or not the ten-and-ones idea of teaching the numbers from 11 to 19 is used, the fact that thirteen means ten and three should be emphasized early in the number program.

When the child has learned to count to 20, the fact that that number means two tens should most certainly be emphasized. The next step is to count by tens to a hundred. If 20 is two tens, it is a logical procedure to ask what is the name for three tens. While the names thirty, forty, and fifty are not quite the same as three tens, four tens, and five tens, there is enough relationship to make the learning of these tens names compara-

tively easy. Sixty, seventy, eighty, and ninety are, of course, so much like six tens, seven tens, eight tens, and nine tens that learning these tens names presents very little difficulty. Few adults and practically no children realize that the syllable "ty" in numbers like sixty means tens. Sixty is then just a short way of saying six tens or the sixth ten.

After the child can count the tens to one hundred, he needs to learn the ones in combination with the tens. He should now learn that the name for the quantity twenty and one more is simply twenty-one, and that in a like manner all the numerals are combined with the ten's name to fill out the series. During the period in which the child is learning to count by ones from twenty to one hundred, the relation between the new use of ones and what the child has already learned about the meaning of these ones should be continually repeated. The tens blocks can be used advantageously to show the relation. For example, 23 can be represented by two tens blocks and three ones blocks. The two tens blocks will appear as two entities just like the first two ones or units that the child learned to call two. And of course the three ones blocks used are identical with the first three learned.

Thus, the major task in counting is to learn the ordinal and cardinal concepts of the first nine numbers. Procedures that emphasize the use of the collection idea (ten as one collection, twelve as one ten and two ones, thirty as three tens), the repetition of facts learned in the first steps in counting (forty as the fourth ten and also as four tens), and the combination of tens and ones (sixty-one, sixty-two, and so on) are illustrations of the proper use of the number system in the teaching of counting. After the numbers to nine have been learned, nothing essentially new is presented, for 10 becomes 1 ten, one collection, and all the other numbers are combinations of this collection and ones, or of its multiples and ones. The child merely uses again what

he has already learned. By means of a slight change, such as the use of tens for ones, different quantities are of course represented. Teaching that tens are collections, and therefore can be dealt with just as were ones, is laying one of the important foundations of later arithmetical work.

The preceding paragraphs emphasize how important it is that children in learning to count acquire a thorough understanding of the first nine numbers. An adequate understanding of these basic numerals is also vital to other areas of arithmetic The teacher should therefore be much concerned with the development of concepts related to the nine digits.

Illustrative Lessons in Counting

Although most of the early counting exercises involve extrinsic motivation, such as saying rhymes and repeating rote counting after adults, the first systematic attempt to teach counting in school should undertake to make use of intrinsic motivation, the kind of motivation which arises from the child's seeing a need for the thing that he is to learn The first lessons that follow are concerned, therefore, primarily with demonstrating to the non-counter a use for counting.

Lesson 1

To begin this lesson the teacher may say· "Susan tells me that she has eight dolls. I wonder how she knows she has that many dolls." Either some child or the teacher will eventually suggest that Susan can find out how many she has by counting. "How do you count? Can anyone show me?" the teacher may ask. If no child tells how, the teacher says, "I count this way: one, two, three, four, five, six, seven, eight." She touches fingers as she says the number names.

"Let's all count fingers. Touch a different finger each time

you say a number word. Count with me." The teacher then touches the fingers on one hand with the index finger of the other, saying one, two, three, four, five, as she does so. She does the same with the other hand beginning with the number six. After several tries at this, the teacher remarks: "The last number name that you use tells you how many fingers you have touched. Let's try it again to see if that's true." Teacher and children then touch fingers, stopping at three. The teacher asks someone what number name was used when the last finger was touched; then asks how many fingers were touched.

Lesson 2

The teacher might begin this lesson by saying: "John says this is the sixth day of school. Is that right? How did he find out that this is the sixth day of school?" If no child offers counting as a solution, the teacher suggests it and then asks for someone to show how to count If no child knows, the teacher says· "I count this way: one, two, three, four, five, six," pointing to fingers as she says the number names. "This, then, is the sixth finger. Now let's try it. Say the number names with me and touch one finger for each name." After trying this on two or three amounts, the teacher says. "I am going to write on the board the names of the days we have been in school. Then we will count them." Teacher and children then count the names of days and find that today is the sixth day.

Teacher then says: "I know another way in which you can use numbers. It's a rhyme most of you know. It goes like this:

One, two, three, four, five,
I caught a hare alive
Six, seven, eight, nine, ten,
I let him go again.

Here's a game I think you will like to play. It, too, uses numbers. 'Here is the beehive.'" Teacher holds up hand with fingers down. "Where are the bees? Here they come. one, two, three," etc. The teacher counts as the fingers of the hand are raised.

Lesson 3

The attention of the children might be directed to number by this statement of the teacher: "The other day when we were playing with numbers several of you counted very well. Who wants to count for us today?" After the volunteers have counted to twenty, the teacher asks some of the others to count, assisting them as they falter, or saying each name for them until they have reached ten. The fingers are used at least one time. The lesson may be ended by using the number games "The Bees" and "One finger keep moving." [1]

Lesson 4

This lesson may be introduced with the following statement: "The first day we counted we found out that it was the sixth day we had been in school. I have the names of the days that we have now been in school marked on the calendar. Is today our ninth or tenth day of school?" After this day (the ninth) has been determined, the teacher asks, "Who can tell how many days we have been in school?" After the correct answer has been given, the teacher remarks, "The last number name not only tells which one, but it also tells how many. Let's count some other things to see if that works."

The lesson closes with an exercise in oral counting to 20 by those who do not already know how to count

[1] Jessie H Bancroft, *Games for the Playground, Home, and School* (New York The Macmillan Company, 1910), pp. 270-72.

Lesson 5

A 20 × 24-inch chart with the numbers to 100 in the following positions is before the children.

	1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99
100									

"This chart has the numbers to 100. Some of you already know how to count this far. Those who do not will learn very soon. All of you know the numbers in the first row. Jack, you say them for us." After Jack gives the names, the teacher points to 10 and asks, "Who knows what this is? Yes, ten is right. Look at it carefully. What is the difference between the 1 in ten and the 1 up here?" (The units 1.) If no child gives the correct answer, the teacher says. "It is no different except there is a zero beside it. This 1 means only one, this 1 means one ten. The zero [points to it] is in the ones place because there are no ones in ten. It is only one ten. Who can find two tens? Yes, that is two tens. What tells you that it is tens and not just two ones?" Again the answer is that zero is in the ones place and, therefore, the two means tens. In a like manner three, four, five, six, seven, eight, and nine tens are located. Attention is then directed to the other names (twenty, thirty, and so on) of these collections of tens. It is pointed out that thirty, forty, and so on, are just easy ways of saying three tens, four tens, and the like. Individual children then take turns counting to 100 by 10's.

Lesson 6

On the day that the abacus is introduced the teacher begins by saying: "Today I want to see how good you are at keeping records. Put on your paper as many marks as the number of fingers I show." The teacher then shows very briefly first 3 fingers, then 2 fingers, then 4, then 2, and then 2. The teacher next asks several different children to tell how many marks they have recorded. The exercise is then repeated (again a total of more than ten marks is needed), but the teacher records the quantity on an abacus. She tells the children who give the correct number of marks that that is what the abacus shows. The teacher also tells the children that the abacus is a counting tool; that it can be used to show numbers (numerically identified quantities) just as marks and numerals are used, that many people once used the abacus for adding sums instead of using numbers; and that many Chinese and Japanese children use it today. She then says, "Let's try one more exercise before we learn how to use the abacus. Perhaps you can see how it is used without further help." This time twelve marks are needed to show the total. A child is asked to put the number of marks needed on the board. Another child is asked to write the quantity, using numbers, and the teacher draws an abacus, marked correctly, on the board. "What does the 2 in your number show?" "What does the 1 show?" "How have I shown the two ones on the abacus?" "How have I shown the one ten?" "What does this row [the ones or right-hand row] of beads stand for?" and "What does the middle row stand for?" are questions asked during the discussion. Each child is then given an abacus. They are told to show 2 ones, 3 ones, and the like, on the abacus; then 1 ten, 3 tens, and the like; then a number like 1 ten and 5 ones. The children are also asked to show these numbers with numerals and marks. In order to add interest to the work, individual children are permitted to represent numbers on their abacus for the others to read. The one reading

correctly the number represented can then present a number on his abacus.[1]

STUDY QUESTIONS

1. Suppose that a six-year-old in trying to enumerate a group of objects says ten as he points to the seventh object. Which of these statements regarding the child's status is sound? (*1*) Although somewhat retarded, the child's number achievement is good. (*2*) The child was probably excited and did not mean to say 10. (*3*) The child has been given a poor start in that the meaning of numbers have been neglected. (*4*) Rote counting has been emphasized too much with this child.

2. In the statement, "He was born in the year 1854," what kind of number is 1854? (*1*) Ordinal. (*2*) Cardinal. (*3*) Both.

3. Is the number twelve an ordinal or a cardinal number? (*1*) Ordinal. (*2*) Cardinal. (*3*) Both.

4. Is counting to find how many a matching process? (*1*) Yes. (*2*) No.

5. Why do teachers object to children's use of counting to find the answers to addition situations? (*1*) Because counting is not adding. (*2*) Because it is inaccurate. (*3*) Because if the child counts he is not likely ever to see a need for adding. (*4*) N.

6. What is the second major step in learning to count? (*1*) Using numbers to find how many. (*2*) Discovering that each number is one more than the other. (*3*) Mastering the order of the number names. (*4*) N.

7. On what grounds may the teaching of rote counting be advocated? (*1*) It gives to the children the meaning of how many. (*2*) It enables children to become familiar with the number names. (*3*) It provides some elementary enumerating experiences. (*4*) N.

[1] For a more detailed discussion on the use of the abacus, see Herbert F. Spitzer, "The Abacus in the Teaching of Arithmetic," *Elementary School Journal, 42*. 448–51 (February, 1942).

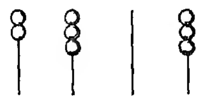
8. Why does the ordinal meaning of number have to be taught before the cardinal? (*1*) It does not. (*2*) Because the mind cannot comprehend how many before you know which one. (*3*) Because order is needed to find how many. (*4*) N.

9. What is the chief disadvantage of the model-group method of telling how many? (*1*) There is no easy way of getting the relation between different quantities. (*2*) It is frequently an inaccurate statement of how many. (*3*) It is difficult to get the meaning or to comprehend the amount expressed. (*4*) N.

10. The tens block is particularly well suited for teaching the meaning of what aspect of number? (*1*) Positional value. (*2*) Notation. (*3*) The base. (*4*) N.

11. For teaching which of these phases of arithmetic is the abacus especially useful? (*1*) Positional values. (*2*) Counting. (*3*) The base. (*4*) N.

12. What number is expressed on this abacus?



(*1*) 3032. (*2*) 233. (*3*) 503. (*4*) N.

13. In reading such a number as 325 should the word "and" be omitted? (*1*) Yes. (*2*) No. (*3*) Depends upon whether current usage or the arithmetic textbook is being followed.

14. What is the major difference between the Roman and the Hindu-Arabic systems of notation? (*1*) One uses figures while the other uses letters. (*2*) The use of a decimal base in the Hindu-Arabic. (*3*) The use of positional value in the Hindu-Arabic. (*4*) The use of fewer different symbols in the Roman.

15. If we used a base of six and used positional value as we now do, how many different numerals or figures would we have to use in writing numbers? (*1*) 5. (*2*) 6. (*3*) 7. (*4*) N.

16. If the number 21 were written to base 7 how many dots would the number represent? (*1*) ....
(*2*) .... (*3*) .... (*4*) N.

17. What people discovered zero? (*1*) Arabs. (*2*) Hindus. (*3*) Babylonians. (*4*) N.

18. In teaching children to count to 100 which of the following should be taught first? (*1*) Count by 1's from 10 to 50. (*2*) Count by 10's from 10 to 50. (*3*) Count by 2's from 10 to 50. (*4*) Count by 5's from 10 to 50.

19. Which of these special advantages can be claimed for a number chart (numbers 1 to 100) on which the first row consists of the first nine numbers (10, the first number on the second row)? (*1*) The positional value aspect of number is emphasized. (*2*) The base idea of number is emphasized. (*3*) The order of number is emphasized. (*4*) The writing of numbers is facilitated.

20. Which of the other names for zero is accepted as correct by teachers of arithmetic? (*1*) Nought. (*2*) Cipher. (*3*) Oh. (*4*) N.

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4

Building Number Concepts

Definition and Character of the Program

In recent years many educators have advocated that a number-concept-building program precede systematic instruction in arithmetic. By systematic instruction is meant the direct teaching of the processes and facts of addition, subtraction, multiplication, and division. A number-concept-building program is not easily defined, because neither the concepts to be developed nor the exercises to be used in developing them have been agreed upon by any representative group of educators. Authorities who advocate a concept-building program merely state that the child needs certain basic concepts before instruction in addition or subtraction can be profitable. What are the concepts prerequisite to addition, is a question that every student of the teaching of arithmetic should ask. More important, however, is a genuine attempt to try to list some of these concepts.

Among the concepts which might be considered prerequisite to addition are: (1) the idea of identifying quantities by means of number names; (2) the idea of grouping to facilitate counting; (3) the idea of substituting one quantity for another; and (4) the idea of rearranging two or more quantities into a single group. Any experience that will facilitate the development of the ideas

just listed would therefore be considered a concept-building exercise. Such procedures are an important part of arithmetical instruction and should receive as much thought as is given to the teaching procedures related to instruction in computational addition.

Careful study of any one of the concepts listed as fundamental to addition will show that no single experience could be expected to develop that concept. For example, consider the idea of applying number names to quantities. The child must learn to apply the name *two* to two fingers, to two rocks, to two plates, and to the second item in a series. The idea of two-ness as applied to all these objects is an abstraction that develops only from many experiences.

From the preceding discussion it can be seen that the concept-building program is an attempt to meet the evident need for meaning and understanding in the learning of arithmetic. The concepts to be built are the ideas that make it possible for the child to see sense in such arithmetical processes as counting, adding, subtracting, and dividing. Adults are so familiar with such ideas as applying the name *two* to any pair of objects that they may make the mistake of taking it for granted that children have the same concepts. To provide experiences that give children an opportunity to learn these "taken for-granted" number concepts is the purpose of the program described in this chapter. The reader should note that the four concepts listed as prerequisite to addition deal with only one of the many areas of arithmetic. Subtraction, multiplication, fractions, measurement, and other phases of arithmetic also require the development of concepts. The reader should also note that many of the suggestions and exercises listed in the chapter on counting make for the development of important number concepts. The most important of those developed in counting are: (1) the series idea of number; (2) the use of counting to find quantity; (3) the relation between the names of ones and the names of tens; and

(4) the collection idea. Close examination of the first two of these concepts developed in counting shows that they contribute directly to the first of the four basic concepts listed as prerequisite for addition.

The suggestion that a concept-building program precede systematic instruction in arithmetic is a bit misleading. The concepts listed in the preceding paragraph and many other number concepts need to be developed as systematically as any other part of the subject. It is, therefore, better to refer to the concept-building program as preceding systematic instruction in learning and applying the basic facts of the four fundamental processes. Since concept-building is to furnish the ideas essential to an understanding of facts and processes, it should be evident that concept-building deserves to be an integral part of instruction in every phase of arithmetic. Because major emphasis on systematic concept-building is found in the primary grades, the program described will be limited to those grades.

The procedures suggested in the following pages are difficult to classify. The title of each exercise identifies one concept, but many of the procedures may be well suited to building a better understanding of several concepts. For example, the counting of a large number of objects may be assigned with the main idea of illustrating to a child the fact that one-by-one counting is an inefficient way to count. Through such experiences, however, he may learn also that that type of counting is not as accurate as group counting, and he may discover the meaning of number words like one hundred one and one hundred two. No attempt will be made to identify all the particular ideas that an exercise may help to develop. The gradation of the exercises is also difficult. Some concepts might be developed in kindergarten, but if these concepts are not developed before Grade Three, the exercises might be suitable at the third-grade level.

Suggested Exercises

1. *Ways of expressing the idea of how many* (Kdg., 1)

"James said that there are only four more days until his birthday. I wonder who can show me with his fingers how many days that is?" One child showed four fingers on one hand. The teacher quickly sketched on the board a hand showing four fingers and then wrote the child's name underneath it. Other combinations were shown and recorded on the board. ||, ||; ||| |; and ||||. The class then took part in a discussion in which the fact that these ways, as well as the written word four and the figure 4, were means of telling the number of days until James's birthday. Counting the number of marks one by one was the usual way of testing the accuracy of a representation, although some children used faster methods, such as counting by groups and adding, "Two and two are four," and "Three and one are four."

This exercise not only exhibited different ways of representing four, but it also illustrated how marks and other symbols are used for days.

2. *Developing the meaning of a number symbol* (Kdg , 1, 2)

"The story we want to read is on page sixteen. In what part of the book will you look for this page?" After the answer, "Sixteen is near the front," had been given, the teacher asked how they knew that sixteen was near the front. "Because you count from the front, and a book has lots of pages," represents the gist of the answers.

After the page was found, one of the children was then asked to tell how he knew that was page 16. When he answered by saying that the "16" meant sixteen, the teacher asked how the book-maker knew that was to be the sixteenth page. The children eventually counted the pages.

3. *Relation between cardinal number names and ordinal idea* (1, 2)

"Who can tell me why I used the number 8 in the date today?" When one child suggested that it meant the 8th day, the teacher asked him to show by pointing to the days on the calendar those that were already past. The other meaning of the 8 in the date, October 8, was then brought out by the following question: "How many days of this month have already passed if you count today?" The answer, "Eight," was volunteered by several.

4. *Counting and developing the idea of time* (1, 2)

The class kept a record of the number of days since a certain event (such as when a ripening tomato first began to change in color) had taken place. Each day an X was placed on that date on the calendar and the children were asked to tell the number of days that had elapsed Sometimes these children were asked to show by marks and figures the number of days.

5. *The ordinal idea of numbers* (2)

"Alice can't find her locker. Her number is 56. Between what two numbers will she find her locker?" When this question was answered, the two pupils who had these numbers were asked to show Alice her locker. Upon their return to the room, the teacher asked Alice if she knew how the class knew that her locker was between those of the two others. The class explanation was something like this: 56 comes after 55 when you count, and 57 comes after 56.

6. *Developing the idea of tens and ones* (1)

"Today let's count the way the men of long ago did. They used their fingers as counters. Ronald, you may be the counter. Let's count the children in the room." After this beginning,

Ronald counted children by matching a finger with a child until he had touched all his fingers. Since there were still more children, another counter was selected to go on with the counting, and Ronald was told to stand aside, but to hold up his ten digits to indicate the number he had counted. A third child was required to finish the counting He used eight of his fingers. The teacher then said, "The men of long ago would have said 'there are as many children as two pairs of hands and eight more.' How many children do you say there are in all?" With a little help, the children decided that each pair of hands was 10 and therefore the total was 2 tens and 8, or 28. The children were then asked to show by marks the number of children in the room. Since none of the children grouped their marks, the teacher complained that it was hard to see whether they had the right number. She told them to try to make their marks so that counting would be easier. Even then the best grouping was not given. After the various suggestions of the children were put on the board, the teacher said: "I'm going to group mine the way men used to do. Here is a group for one pair of hands; here is a group for another pair, and here are the eight ones. Now which of these methods shows twenty-eight most clearly?" The grouping into two tens and eight ones was, of course, by far the most simple.

7. *Relation between number names, numerals, and quantities* (1, 2)

In answering the question, "How many more days until Hallowe'en?" Tommy wrote the word *seven*. What other ways might Tommy have used to show the number of days until Hallowe'en? The following ways were suggested: 7, IIIIIII, five fingers and two fingers, and OOOO OOO. The teacher emphasized the five-finger and two-finger arrangement and said there are other ways of showing seven The children were then told to show seven with sticks in as many ways as they could.

8. *Substitution of one quantity for another* (1, 2, 3)

"In Bruce's statement, 'The tomatoes in our garden are high,' we don't really tell people much about the height of the tomatoes, do we? Let's try to think of a way of saying this so that people will know what we mean."

Among the many suggestions the following were accepted by the class as being worth while:

(*a*) The tomatoes are as high as the table.
(*b*) The tomatoes are as high as my desk.
(*c*) The tomatoes are as high as Ann's shoulder.
(*d*) The tomatoes are as high as my arm is long.

At this time the teacher made no attempt to introduce a standard measure. Procedures of the type used by the children are primarily to promote growth of the basic idea of measurement; namely, that another thing can be used as a substitute for a needed dimension of the real thing.

9. *Use of measures to make for easy comparison of quantities* (2, 3)

In a discussion period Tim and Jane got into an argument regarding the size of their rabbits. The class attempted to help settle the argument. First, points to be considered in deciding size were treated. From big and bigger, specific points like length, height, and later weight were considered. Since the height and the length of a rabbit are difficult to get, the group instructed Tim and Jane to get someone to help them weigh their rabbits.

The next day after the weights had been reported, different children were asked to find one or more books that weighed just what each rabbit weighed. The books representing the weight of the rabbits were then compared by lifting and by looking at the size of each. The fact was brought out that size according to sight showed the rabbits to be about the same size, but that

a scale showed that Jane's rabbit weighed a little bit more than Tim's.

This exercise, too, is designed to promote the idea of substitution in measurement.

10. *Exercises to show the sense of using numerals* (1, 2)

The names of children absent from the room were written on the blackboard. The teacher asked how else she might have shown that there were that many people absent. Suggestions were: (*a*) to make a mark for each child absent, (*b*) to put all the empty desks together; (*c*) to write the figure 11; (*d*) to write the word eleven; and (*e*) to show eleven sticks. (These sticks had frequently been used to represent quantity.) The children were then asked to indicate which of these ways showed most clearly the number of children absent. It was decided that the marks and sticks were better than the names and desks because you could see the marks and sticks more easily. Some children thought the figure 11 was best. The teacher agreed, but only if they really knew what the 11 meant. She then asked what 11 meant. The answer "1 ten and 1 one" was given. The teacher then showed the class a tens block (see illustration, page 79) in which sticks the same size as those used in showing eleven were partially sawed out. After a brief explanation of the tens block, the teacher asked if someone would show what 11 means by using the tens block and the sticks or ones block. When 11 was represented in this way, the children all agreed that the numerals provided the best way to record the number of children absent. "By using the figure 11, you have to write only two things: one to show the ten and one to show the one," was the teacher's summarizing statement.

Other quantities like 21, 22, and 52 were then shown objectively by using both tens blocks and ones blocks and by using the ones blocks alone.

11. *Rearrangement of numbers to represent the same quantity* (2)

As a part of an oral exercise in the second grade, children were asked to tell how they arrived at certain sums and remainders. Many different methods were revealed. For example, one child said he knew that 3 and 4 equal 7 because 3 and 3 made 6 and 4 was one more than 3; therefore, the answer was one more than 6. In dealing with sums greater than 10, the method of making tens and ones was illustrated in 8 plus 5. The child said: "I took 2 from the 5 and put it with the 8 to make 10. Then there were 13 in all."

12. *Meaning of twice* (1, 2)

"Jean says her overshoes are twice as heavy as her shoes. How can we find out whether she is right?" After some discussion in which both the method to be used and the meaning of twice were considered, the teacher suggested that they weigh Jean's overshoes and her regular shoes. The two shoes weighed a little more than one overshoe. In the discussion, the teacher stated that if one overshoe weighed as much as two regular shoes, then the pair of overshoes must weigh twice (two times) as much as the pair of regular shoes. The children then engaged in a weighing experience to find two objects that weighed the same or one of which weighed twice, three times, or four times as much as the other. The weight evaluation was made by comparing the distance that the marker on the scales moved.

13. *The meaning of tens numbers* (2, 3)

Toward the end of an oral exercise in the second grade, some examples like 20 plus 30 were given to challenge the better pupils. Those children who gave immediate answers were asked to tell how they thought. The explanations either involved knowledge of 20 and 30 or a counting by 10's as in 30, 40, 50. The teacher then asked the class for another way of saying 20 (2 tens) and 30 (3 tens). She then asked how much were 2 tens and

3 tens. She then asked for another way of saying the answer 5 tens (obviously fifty).

14. *Using a number chart* (1, 2)

Each of the following directions refers to the number chart (page 94): (*a*) Find the number that means 2 tens and 3 ones. (*b*) Find the number that is 10 more than 2 tens and 3 ones. (*c*) Find the sixth ten. (*d*) What number comes just before 67? 33? (*e*) What number is 3 tens larger than 36? (*f*) Show the row of numbers with only 3 tens and ones in it.

15. *Thinking of combinations that equal a certain number* (2, 3)

The teacher began the lesson by saying. "I'm thinking of two numbers that make 8. Can you tell what the numbers are?" Children offered different combinations such as 4 and 4, 6 and 2, and 7 and 1. The teacher did not accept the first combinations. She said: "Those numbers do make eight, but I'm thinking of two other numbers that make 8." After a combination for 8 was accepted, the teacher thought of other numbers. When a wrong combination like 7 and 5 for 11 was proposed, the teacher asked the child who had given that incorrect combination to go to the board and show by means of marks and counting that 7 and 5 equal 11. The teacher then asked the child that had given the last correct response to take the teacher's place. After that, each child giving the accepted correct response was permitted to suggest a number.

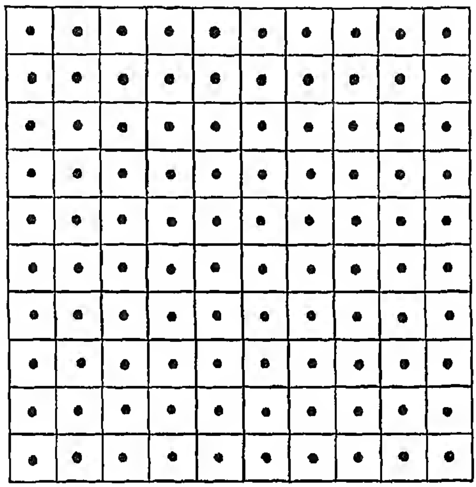
This exercise can be varied to include three numbers and even four numbers can be used occasionally. Teachers will find that if children suggest the numbers, certain rules (for example, no numbers larger than 18) must be set up. It is well for the teacher to offer combinations when a child has suggested a particularly difficult number.

The exercises may also be changed to require subtraction. In order to avoid having only one possible correct answer, the

teacher would say, "I'm thinking of two numbers that have a difference of 4."

16. *Using the tens square* (1)

The lesson was introduced by passing out tens squares to all of the children.

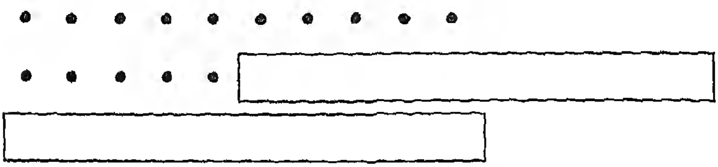


The tens square was about 8 inches on a side. Every child was also given two cardboard markers, each the length of one side of the square and about the width of one row of squares.

The children were asked, "How many dots do you think there are on your square?" Several children volunteered estimates, which were written on the blackboard. Then the teacher asked, "How can we find which of these numbers is correct?" The children suggested counting. They counted the dots and discovered there were 100 in the square.

The teacher then asked several children how they counted, and received the answer that they had counted by ones. The question was then asked of a little boy who had counted very

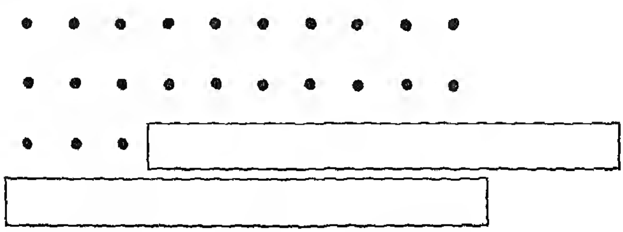
rapidly. He replied that he had counted by tens. When asked to show the children how he did this, he placed his marker under each row of 10 dots, saying, as he did so, "Ten, twenty, thirty," and so on. The children all counted their squares by tens and discovered there were 10 rows of 10 on the square. They also discovered that there were 10 columns of 10 each. The teacher then placed the marker on her number square thus:



"I have made a picture of the number 15 on my number square, for I have 10 dots and 5 more. Can you make a picture of 15, too?"

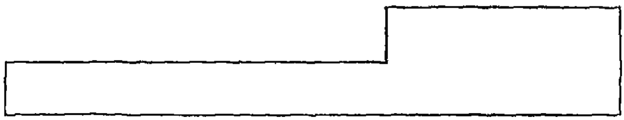
After this had been done, the children were asked to show the following numbers on the square: 21; 50; 9; 77; 100. In commenting about the various amounts shown, the teacher brought out the fact that tens and ones were shown. For example, for the 21, the teacher said, "Yes, that's right because here are your two tens, and there's your one. That makes twenty-one."

The lesson was concluded by asking the children to show any number they wished, up to 100, on the number square. Each showed his square to the whole group and told which number he had shown, thus:



"I have a picture of 23, for I have 2 tens and 3 more."

Instead of two markers a single long marker of the design shown below may be used.



17. *Showing what numbers mean* (1)

This lesson was introduced by the teacher's asking the children to show the number 5 with the fingers of one hand. One child then went to the blackboard and drew marks to show the way his hand had looked. The children counted the marks to make certain that there were five of them.

Then the teacher had the children show 5, using both hands. The children did not all show 5 in the same way. Some showed 2 fingers on one hand and 3 on the other; one showed 4 fingers on one hand and 1 on the other, and so on. As the teacher noted these different combinations of 5, she had the children go to the blackboard and draw pictures of the way their fingers had looked. The following arrangements of marks were used:

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  |||    ||
 ||||     |
|||||
   ||   |||
    |  ||||
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Then the teacher said, pointing to the first group of marks: "This is the way Tom's hands looked. Let us count and see if he has shown 5." One child touched each mark as all counted. The other groups of marks were similarly counted. Since the emphasis in this lesson was on counting as a method of checking, the teacher did not emphasize the fact that "2 and 3 are 5," "4 and 1 are 5," and so on. Other groups of marks were similarly counted. It should be noted that *at no time* did the teacher

say, "2 and 3 are 5." The emphasis was placed entirely upon counting as a method of checking.

The children were each given five toothpicks. They were asked to count them and then to arrange them in two groups. Various groupings were given and "pictures" of these, using circles, were drawn on the blackboard.

```
OOO     OO
  O     OOOO
 OO     OOO
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One child was permitted to check each "picture" to determine whether or not it was correct. The lesson was concluded by having the children use paper and pencil and show the number 5 in as many ways as they could.

18. *The meaning of zero* (2, 3)

"Several times I have asked different members of the class to show what figures like 6 and 8 mean. I wonder if anyone can tell me what the figure 0 (zero) means? Does it stand for a quantity as the other figures do?"

A brief discussion following this introduction revealed that many of the children were not certain about the meaning of zero. The teacher then suggested that they write the symbols 1, 2, and 3, and put marks underneath each symbol to show what it means. She then asked that they write the symbol zero and then show what it means. Of course, the children could not write anything to show what zero means. The only place where zero can have any meaning is as a place-holder; therefore, zero cannot be used alone to show meaning or function. Children who tried to show the meaning of zero by writing it alone were challenged by questions like this: "If there is no quantity to write, why write at all?" Children who used zero in numbers like 10 and 20 had little difficulty in telling that zero held a place and thereby made it possible for a person to be able to tell that the 1 and the 2 referred to tens. The children who did not

immediately see this use of zero were directed to write underneath the 1 and the 2 the amounts they represented. They were then asked to explain why the 1 in 10 meant ten, whereas in a drawing done earlier in the period it meant only one. That is, how did they know that they should read one 1 as a one and another 1 as a ten? In order to emphasize this point the teacher directed the attention of the entire class to the number chart (see page 94). She asked different children to find two ones on the chart, then two tens, and so on. Those who performed this assignment successfully were asked to tell others how they knew that this particular 2 meant two tens. After these points were fully discussed, the teacher asked, "Then what does the zero do?" It was decided that holding a place was its real job.

19. *The advantage of a place-holder* (2, 3)

Soon after the preceding lesson on the meaning of zero, another lesson was opened with the following remarks: "Many people did not use a zero, a place-holder, in writing numbers the way we do. The Roman numerals, as you well know, do not have a zero. Do you know what advantage there is in our way of writing where we use a place-holder? Let's write the Roman numbers to 12 to see if we can see any advantage."

The class with the aid of the teacher then wrote the Roman numbers to 12 on the board. Just above each Roman number the teacher placed the corresponding Hindu-Arabic.

"Now let's see. What do we use to show ten?"

"A one and a place-holder or zero," was the accepted answer.

"And what did the Romans use?"

"They used an 'X,'" was the children's reply. To that the teacher added that the "X" was a new symbol and used only for tens. She then asked for someone to write thirty, using Roman numerals and Hindu-Arabic. This was followed by the writing of fifty, seventy, and one hundred. The teacher, of course, wrote or helped with the last three numbers. She

then directed the attention of the class to the essential difference between the two schemes by questions like the following: "What did we use in writing thirty that we had already used before?" "What did the Romans use?" "How did we show that we were dealing with thirty or three tens?" "How did the Romans do it?" The same type of questions was used for fifty, seventy, and one hundred. Following the showing of these differences, the teacher asked this final question. "What, then, is the advantage of using a place-holder?" The answer is, of course, that with the Hindu-Arabic system a person can write any number no matter how large as soon as he learns to use the ten symbols.

20. *Learning what a fraction means* (1, 2, 3)

"Sally, will you get this cup about one-half full of water?" asked the teacher as she and some of the children worked at the table preparing tomatoes to be cooked. When Sally returned with the glass half full, the teacher stopped and asked the children if Sally had the right amount. When one boy did not know, the teacher asked another to explain.

The explanation was something like this: "When it is full it is up to here. Half full would be only halfway up to here. It's half full and half empty."

"That's right," said the teacher. "The part that is empty is just as much as the part that is filled with water."

Other exercises involving the concept of one-half grew out of the need for boards one-half as long or half as thick, an object one-half as heavy, one-half as tall, one-half as many books. Of course, many of these situations can only result from careful planning by the teacher. She must be on the alert for concept-building occasions.

21. *Number games*

A number of games which children play are excellent for building number concepts. A few of the more desirable and popular of these games are listed below.

(*a*) *Dominoes.* One of the simplest games played with dominoes involves simply matching the dominoes correctly until all dominoes have been used. The object of the game is merely to play all the dominoes. From one to four players may participate in this game. In a variation of the above (involving two to four players) the object is to be the first to get rid of all dominoes. A further refinement makes getting the highest score the object. Scores are obtained either by counting the largest number of dots that appear on any one domino or by counting only fives or multiples of fives. These games of dominoes develop the ability to see groups without counting, to match, to count, and to add. The double six set of dominoes will be found simpler than the double nine.

(*b*) *Hide and seek.* In this game one child is the seeker while the others hide. The time permitted for hiding is determined by the length of time it takes the seeker to count to 10, 20, or some other number by ones, or to count to 100 by fives. Major concepts developed are the order of number names and a feeling of the place of numbers in a series. For example, the child who is to hide soon learns that when the seeker says 85 or 90, it is time to be hidden.

(*c*) *Number authors.* A pack of cards using only numbers through 6, 7, 8, or 9 is used. Each child is dealt a specified number of cards. On each card is one number. Players take turns asking for a card from others. If the player asked does not have the card, the asker draws from the deck. The object of the game may be to get pairs of numbers or to get cards to total 10 (or other number) in value.

(*d*) *Twinks.* This game is played with a deck similar to that for "number authors," but all cards are dealt. Play proceeds by having each player in turn place a card face up before him. The object of the game is for a player to get a group of cards showing numbers that total 10 (or any other number). As soon as this happens the person who first says "Twinks" gets the

cards that total 10. The child having the most cards when all cards have been played is the winner.

(*e*) *Lotto, keno, or bingo.* This game is played with rectangular boards on which are numbered squares. When a number is called which appears on a player's board, he covers that number with a marker. The player who first gets a row, column, or diagonal of squares covered wins the game. Recognition of numbers is probably the most important thing developed by this activity.

(*f*) *Bean-bag.* To play this game a canvas bag containing beans and a section of floor with numbered partitions are needed. Children take turns tossing the bag on the numbered section of the floor. The object is to have the bag stop on the partition marked with the largest number. There are many variations of this game. It may be played by individuals or by teams.

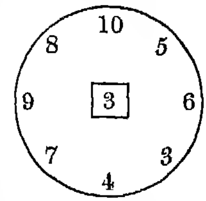
An important thing to note about these games is the fact that each furnishes the child a lifelike setting for the use of numbers. The child plays the game for the purpose of winning it from his competitors. The use of numbers or number ideas is an integral part of the game.

22. *Number devices*

Teachers of arithmetic have devised a great many game-like situations for the purpose of teaching various phases of arithmetic. These devices differ from the games described above in that children do not engage in them in order to win by doing better than other players. A few of the devices that are of value in developing number concepts are listed below. Care should be exercised in the uses of such devices, for they may easily become merely a disguised form of drill.

(*a*) *The circle game or play teacher.* A circle such as the one shown is used. One child plays the part of the teacher, pointing to various numbers on the circumference, and calling upon different classmates to give the sum or difference between the number

pointed to and the number in the center. There are many variations of this device. It makes for familiarity with sums and differences and gives the child a better idea of the combination of numbers and of the composite nature of numbers.



(*b*) *Placing people in houses.* The teacher draws three houses on the board stating that eight people live in the three. She asks different pupils to show (first with marks and later with numbers) the number of people that live in each house

(*c*) *Climbing the stairs to the fairy's castle.* In one form of this device a series of steps leading to a castle is pictured. On each step is a number or group of numbers. The child who is able to read each number on a step is permitted to go the next step and if able to do all is said to reach the fairy's castle.

23. *Systematic exercises*

There is a large number of systematic exercises which undoubtedly aid in the development of number concepts. Examples of these are listed below. One concept related to each exercise is mentioned.

(*a*) Draw a circle around the word that tells how many dolls there are in the picture. One concept: Counting to find a number.

(*b*) Make as many marks as there are rabbits in the cage. Write the number that tells how many rabbits there are. One concept: substituting one quantity for another.

(*c*) Draw four windows on the house. One concept: The cardinal idea of four.

(*d*) Put a circle around the box that has the most dots in it. One concept: Recognition of use of counting numbers to find most.

(*e*) Make as many dots in the circle as there are in both the boxes. One concept: Equivalence between two quantities and a single quantity.

(*f*) Color four of the balls green. One concept: Recognition of use of ordinal series to find how many

(*g*) Write the missing numbers in the blanks. One concept: The position of numbers in the series.

(*h*) Color the second ball red. One concept: The ordinal idea of number.

(*i*) Connect the numbers in the correct order and find out what animal is hidden on this sheet of paper. One concept: The ordinal idea of number.

STUDY QUESTIONS

1. In counting a large number of objects (e.g., 150) what number concepts will the child acquire? (*1*) The meaning of number names above 100. (*2*) That one-by-one counting is likely to be inaccurate. (*3*) That group counting is more efficient than one-by-one counting. (*4*) The concepts which children might acquire cannot be identified with much assurance.

2. The tens square (a square of 100 dots or squares, 10 on a side) is especially suited for the development of what number ideas? (*1*) That ten tens equal 100. (*2*) The meaning of 100. (*3*) The meaning of the tens numbers. (*4*) N.

3. What distinct advantage does the exercise in which children guess the number others are thinking (e.g., "I am thinking of two numbers that equal 9") have over the exercise in which the child is asked the sum of two numbers such as 7 and 2? (*1*) The approach is from the sum, the natural way to study numbers. (*2*) Since the sum is already given, the child sees the whole fact. (*3*) It requires the use of both addition and subtraction. (*4*) N.

4. Number games are often very useful in teaching arithmetic. What is it that games provide? (*1*) Lifelike settings for the

use of numbers. (*2*) Practice which is painless. (*3*) Excellent illustrations of the meaning of numbers. (*4*) N.

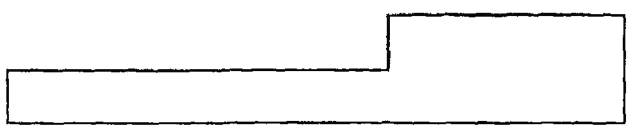
5. When children engage in such activities as giving the sums of numbers as a means of climbing the stairs to a "fairy's castle," is the motivation intrinsic or extrinsic? (*1*) Intrinsic. (*2*) Extrinsic.

6. In the exercise in which the teacher says, "Who can tell me why I used the number 11 in the date for today?" what concept is most likely to be developed? (*1*) The cardinal meaning of 11. (*2*) The ordinal meaning of 11. (*3*) That cardinal and ordinal are closely related.

7. Should the teacher emphasize by stating verbally the concept that children are to acquire through use of a given exercise? (*1*) Yes. (*2*) No.

8. Could the number chart be used to help children in learning to read such numbers as 17 or 43? (*1*) Yes. (*2*) No.

9. In the exercise with a tens square where two cardboard markers are used to show such numbers as 23, would there be any advantage in using only one marker of this type:
(*1*) Yes. (*2*) No.



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5

Addition and Subtraction

THE TRUE NATURE OF THE FUNDAMENTAL PROCESSES

The four fundamental processes — addition, subtraction, multiplication, and division, with whole numbers and fractions — comprise the major part of present-day arithmetical content. Since the chief contribution of arithmetic is in the simplifying of concepts, it follows that the purpose of these processes is one of simplification. In Chapter 1 it was demonstrated how one of these processes (addition) is used to simplify (see pages 3 ff.). In the situation described, a group of 6 ones and a group of 8 ones were regrouped into 1 ten and 4 ones. Thus addition is really only the regrouping of two or more groups into a single group, or into groups of tens and ones. The simplicity of this rearrangement of two or more groups into one group or into groups of tens and ones can be demonstrated visually through the use of the tens blocks in such an exercise as the following: Add $7 + 4 + 6 + 5$.

Subtraction, too, involves regrouping. In this case, however, a single number or group is divided into two groups, the size of one of these two groups being known. Multiplication and

division are also simply forms of regrouping in order to express more clearly the desired quantitative situation.[1]

The fact that these processes are merely accepted ways of regrouping or rearranging quantities cannot be overemphasized. Far too many who have studied arithmetic think of the addition and multiplication processes as something done to numbers to "get more," and of subtraction and division as processes which "give less." As an interesting check on the foregoing statement, ask several of your friends whether multiplication of two or more numbers means "making more." The answer will usually be yes. Yet, by definition, it cannot mean more. Four threes equal twelve. They are not more than twelve.[2] To say that the fundamental processes merely give regroupings is not in any way to detract from their importance. Regrouping always results in simplifying the numerical situation by stating the result in standard terms, thereby making easier any further thought about the quantities in question.

Beginning Systematic Instruction in Addition and Subtraction

In the preceding chapters most of the simple examples and most of the exercises were concerned with addition. The reader should not infer from this fact that instruction in addition should precede instruction in subtraction. The author of this book recommends that the two processes be developed simultaneously. Because they are complementary processes, the learning of the two together should aid the understanding of each. That more references have been made to addition is merely a reflection of the fact that in life more addition than subtraction occurs.

[1] Harry G. Wheat, *The Psychology and Teaching of Arithmetic* (Boston: D. C. Heath and Company, 1937), p. 132.

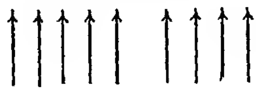
[2] Roy Edgar Adams, *A Study of the Comparative Value of Two Methods of Improving Problem-Solving* (Thesis, University of Pennsylvania, 1930), pp. 60–61.

In a preceding paragraph, addition and subtraction as processes of regrouping were stressed. To this idea of the processes should be appended their relation to counting. In the counting exercises and in the exercises for building number concepts, many of the basic ideas of both processes were learned. For example, in exercise 17, page 111, where different ways of showing 5 are considered, the child can see that when one of five objects is moved away from the group, four remain (:·:   : :   .); that $4 + 1 = 5$; $5 - 2 = 3$ (:·:   ..   ...); $3 + 2 = 5$, and so on. Likewise in the type of exercise in which the child is to find the group of objects that has two less than another group, he can learn that $8 - 2 = 6$.

While exercises and experiences of the type referred to in Chapters 3 and 4 are prerequisite to systematic instruction in addition and subtraction, they cannot of themselves be depended upon to insure the child's learning of these two processes. Note that the term *prerequisite* was used. A good foundation in counting is essential to efficient learning of the processes. This does not mean that some arbitrarily established standard in counting must be reached before any work in addition and subtraction is to be undertaken. To prevent such rigid compartmentalization of number instruction, the various exercises in Chapters 3 and 4 were suggested.

After the child has had experience of the type referred to, exercises designed primarily to promote addition and subtraction should be used. Such exercises can be easily introduced in the oral arithmetic period. For example, the teacher may say: "I see a number on the chart that is 3 less than 5; or 2 more than 6; or 10 more than 30." (See lesson 1, Chapter 14.) The following are other ways of centering attention on the processes. "Who can tell me how much 4 and 4 are? 6 and 2? 4 take away 1?" and so on. Frequently children may be asked to show with blocks or other objects or by marks on the blackboard that their answers are correct.

Problems stated orally can be used advantageously in these initial exercises. The chief advantage of a problem is to furnish a setting that children can visualize or otherwise imagine. Either the children or the teacher can diagram such a problem on the board. For example, the problem, "Yesterday we had no arrowheads. Today Nancy brought five and Jane brought four. How many do we now have?" was illustrated in this way by members of a second-grade class.



The following are examples of other exercises which may play an important part at this stage of instruction:

1. Each child is given a number card (one to nine marks on one side and the numerals 1 to 9 on the other). One child is appointed leader. He takes his place at the front of the room and invites another child to come up. These two show their cards to the class. The leader then calls on some volunteer to give the sum or the difference of the numbers, whichever process is being emphasized. If the child gives the correct answer, he comes to the front of the room. The first leader goes to his seat while the child already at the front becomes leader. At first only marks are used. Then numbers and marks are mixed. Finally only numbers are used. Later, in addition practice, three children may appear together at the front.

2. The teacher asks the children to add a number such as 3, 4, or 8 to the number of fingers shown; or she asks them to add to a specified number such as 8 the number of fingers shown.

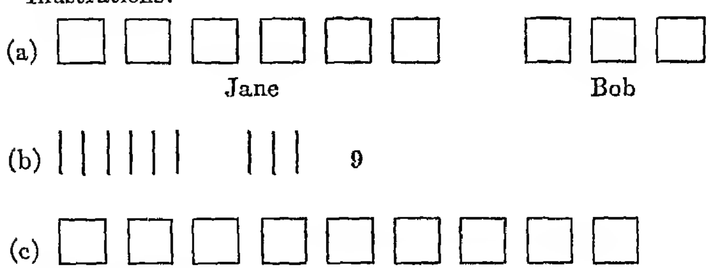
3. The teacher asks the children to find the number of objects in two boxes; to find the box that has two fewer objects than in the circle she puts on the board.

After foundation work of the type described above, the systematic program of instruction may be started in the following

way: Problems involving addition and subtraction facts are written on the board or those found in a textbook are considered. In either case the teacher reads the problems aloud. In this way the usual handicap under which the poor reader operates will be avoided. The directions given may be something like the following: "Instead of asking questions today and letting just one of you at a time answer, I have written several problems on the board. You are first to write just the answer to the question in each problem. If you do not know the answer, draw a picture showing what the problem tells and then try to answer the question. If you would rather use the actual objects to solve the problem, you may do that." Those who solve the problem without a diagram are told to prove that their answer is correct by drawing diagrams or marks. Later they are asked to show with numbers or marks a record of how they thought. A sample of the illustrations or diagrams and the record of thinking made by children in working two problems appear below:

Problem 1. Jane read the first six pages of a book and Bob read the next three. How many pages did the two read?

Illustrations:



Number record:

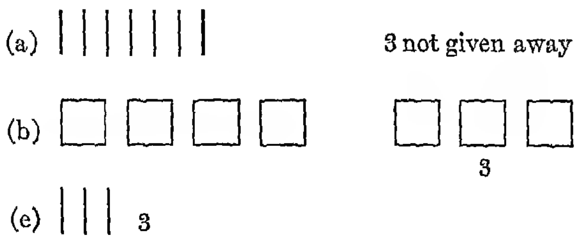
(*a*) 6 and 3 make 9

(*b*)
$$\begin{array}{r} 6 \\ +\ 3 \\ \hline 9 \end{array}$$

(c) 6 and 3 = 9

Problem 2. Four of the seven new books have already been given to children. How many have not yet been given away?

Illustrations:



Number record:

(*a*) 7 take away 4 is 3

(*b*)
$$\begin{array}{r} 7 \\ -\,4 \\ \hline 3 \end{array}$$

Examples of these various records should be put on the board and the children asked to decide whether or not the record agrees with what the problem states, and whether or not the record is clear enough to enable others to tell how the pupil thought. The latter can be determined by asking pupils other than the one who made the record to tell how the thinking was done. Such a procedure puts a premium on clear records of thought. If the record of thinking is clear but the pupil asked to read the record fails to understand it, the teacher may say, "I believe I see how it was done. You just don't see it at this moment; perhaps that is because you don't think about the problem in the same way as the person did who made this record. Let's see if his reading of his record will make it clear to you?" The person who has made the record then reads it and others ask questions if there are points they do not understand.

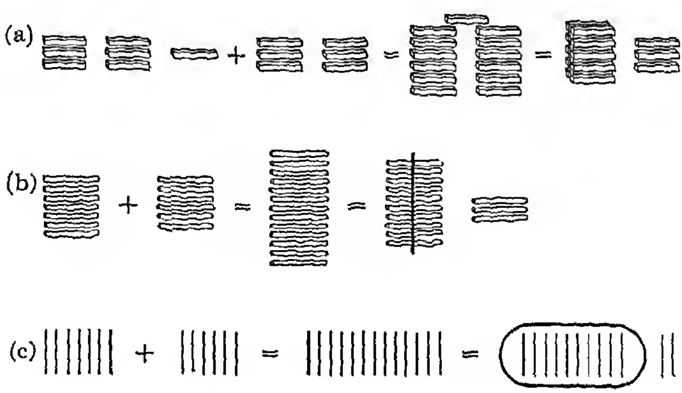
As indicated by the above description, the teacher should take an active part in guiding the evaluation, but she should

not hurry the children to some generalization they do not understand. For that reason every correct method should be accepted, but eventually the discussion should be directed toward the selection of the best method. Although this best one should, of course, be recommended to the class, it should not become immediately the only accepted procedure. The various records of thinking, many of which will be crude, furnish the child some basis for accepting the short adult record as the superior one. In order to help those who are having special difficulty making and reading records of thought, the actual manipulation of objects such as pages and books should be permitted or required. Some of the better students can also be asked to do such manipulation. This procedure will demonstrate to them more forcefully the economy in writing, reading, and thinking that results when the shortest arithmetical procedures are employed.

In connection with many of these problems the idea of regrouping should be illustrated For example, in a problem calling for the addition of four sticks and three sticks, to find how many altogether, the children should be asked to do what the problem suggests; that is, put them all together. Then the question, "How many more sticks are there now than when you had them in two groups?" should be asked. The children will see that no new sticks were added and that the quantity is therefore the same. The next question, "Does addition mean 'more'?" would be as easily answered by the children. Time should also be taken to discuss the reason for regrouping into only one group. The fact that one group (number) is easier to say, to write, and to think with than are two groups (numbers) is the primary reason for the regrouping (addition). Another and probably equally important reason for regrouping in such situations is that in common usage that is the only procedure employed. The simplifying effect of regrouping quantities into standard groups can be more easily demonstrated if sums larger than 10 are used and if the tens block is used.

That the process of subtraction means the separation of one specified group from a larger group is made obvious in a real problem situation such as the following: "Six of the eleven dolls belong to the second-grade room. The others are Mary's dolls. How many dolls does Mary have?" To show objectively the answer to the question, it is necessary to separate the six dolls belonging to the room from the entire group of eleven. In this way the dolls belonging to Mary may be identified and enumerated.

In problems involving addition facts with sums above nine, the children should be required to show frequently that in adding two or more numbers they are rearranging the numbers into groups of tens, or of tens and ones. The tens block is a good device for demonstrating this outstanding characteristic of number. Sticks tied into bundles of ten can also be used to advantage. The circling of marks on the board or on paper to segregate groups of ten is a desirable procedure for semi-concrete representation. Each of the three methods of showing that addition is a matter of rearranging into tens and ones is shown below. The addition fact $7 + 6 = 13$ is illustrated (*a*) with a tens block and ones blocks, (*b*) with bundles of sticks, and (*c*) by circling marks.



In each illustration the last arrangement of one ten and three ones stands out sharply when compared with a single group of thirteen ones.

Basic Addition and Subtraction Facts

All the basic addition and subtraction facts, with perhaps the exception of the very easy ones, should be demonstrated in the manner described in the preceding paragraphs. By basic addition facts are meant the combination of each of the numbers from 1 to 9 with every other one-figure number and with the number itself.

The basic addition facts are given on page 130. The facts with sums of 10 or less are generally referred to as the easy facts.

The basic subtraction facts are given in the accompanying table (page 131) and are, of course, the reverse of the addition facts. The facts with minuends of 10 or less are usually referred to as the easy subtraction facts.

Attention is called to the fact that no so-called "zero facts" are included in either the list of addition facts or the list of subtraction facts. The child has no need for the addition zero facts until he undertakes the addition of two-place numbers. Likewise, the zero subtraction facts are not needed until the subtraction of two-place numbers is begun. It is therefore recommended that the teaching of the zero addition and subtraction facts be postponed until after a use for them has arisen in work with two-place numbers Such postponement is not only in harmony with the belief that the child shall see sense in what he does, but it might prevent some of the misconceptions regarding zero that are now fostered in attempts to teach zero facts meaningfully. For example, one course of study [1] advocates that the pupil be "kept clear in the concept of zero by teaching that it means nothing, none, not anything . . . develop

[1] *Mathematics Curriculum,* Watertown Elementary School, Watertown, New York, 1936. See first and fifth pages of First Grade Outline.

81 ADDITION FACTS

1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	10
2	2	2	2	2	2	2	2	2
1	2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10	11
3	3	3	3	3	3	3	3	3
1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	10	11	12
4	4	4	4	4	4	4	4	4
1	2	3	4	5	6	7	8	9
5	6	7	8	9	10	11	12	13
5	5	5	5	5	5	5	5	5
1	2	3	4	5	6	7	8	9
6	7	8	9	10	11	12	13	14
6	6	6	6	6	6	6	6	6
1	2	3	4	5	6	7	8	9
7	8	9	10	11	12	13	14	15
7	7	7	7	7	7	7	7	7
1	2	3	4	5	6	7	8	9
8	9	10	11	12	13	14	15	16
8	8	8	8	8	8	8	8	8
1	2	3	4	5	6	7	8	9
9	10	11	12	13	14	15	16	17
9	9	9	9	9	9	9	9	9
1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18

81 SUBTRACTION FACTS

$\begin{array}{r}2\\1\\\hline 1\end{array}$	$\begin{array}{r}3\\1\\\hline 2\end{array}$	$\begin{array}{r}4\\1\\\hline 3\end{array}$	$\begin{array}{r}5\\1\\\hline 4\end{array}$	$\begin{array}{r}6\\1\\\hline 5\end{array}$	$\begin{array}{r}7\\1\\\hline 6\end{array}$	$\begin{array}{r}8\\1\\\hline 7\end{array}$	$\begin{array}{r}9\\1\\\hline 8\end{array}$	$\begin{array}{r}10\\1\\\hline 9\end{array}$
$\begin{array}{r}3\\2\\\hline 1\end{array}$	$\begin{array}{r}4\\2\\\hline 2\end{array}$	$\begin{array}{r}5\\2\\\hline 3\end{array}$	$\begin{array}{r}6\\2\\\hline 4\end{array}$	$\begin{array}{r}7\\2\\\hline 5\end{array}$	$\begin{array}{r}8\\2\\\hline 6\end{array}$	$\begin{array}{r}9\\2\\\hline 7\end{array}$	$\begin{array}{r}10\\2\\\hline 8\end{array}$	$\begin{array}{r}11\\2\\\hline 9\end{array}$
$\begin{array}{r}4\\3\\\hline 1\end{array}$	$\begin{array}{r}5\\3\\\hline 2\end{array}$	$\begin{array}{r}6\\3\\\hline 3\end{array}$	$\begin{array}{r}7\\3\\\hline 4\end{array}$	$\begin{array}{r}8\\3\\\hline 5\end{array}$	$\begin{array}{r}9\\3\\\hline 6\end{array}$	$\begin{array}{r}10\\3\\\hline 7\end{array}$	$\begin{array}{r}11\\3\\\hline 8\end{array}$	$\begin{array}{r}12\\3\\\hline 9\end{array}$
$\begin{array}{r}5\\4\\\hline 1\end{array}$	$\begin{array}{r}6\\4\\\hline 2\end{array}$	$\begin{array}{r}7\\4\\\hline 3\end{array}$	$\begin{array}{r}8\\4\\\hline 4\end{array}$	$\begin{array}{r}9\\4\\\hline 5\end{array}$	$\begin{array}{r}10\\4\\\hline 6\end{array}$	$\begin{array}{r}11\\4\\\hline 7\end{array}$	$\begin{array}{r}12\\4\\\hline 8\end{array}$	$\begin{array}{r}13\\4\\\hline 9\end{array}$
$\begin{array}{r}6\\5\\\hline 1\end{array}$	$\begin{array}{r}7\\5\\\hline 2\end{array}$	$\begin{array}{r}8\\5\\\hline 3\end{array}$	$\begin{array}{r}9\\5\\\hline 4\end{array}$	$\begin{array}{r}10\\5\\\hline 5\end{array}$	$\begin{array}{r}11\\5\\\hline 6\end{array}$	$\begin{array}{r}12\\5\\\hline 7\end{array}$	$\begin{array}{r}13\\5\\\hline 8\end{array}$	$\begin{array}{r}14\\5\\\hline 9\end{array}$
$\begin{array}{r}7\\6\\\hline 1\end{array}$	$\begin{array}{r}8\\6\\\hline 2\end{array}$	$\begin{array}{r}9\\6\\\hline 3\end{array}$	$\begin{array}{r}10\\6\\\hline 4\end{array}$	$\begin{array}{r}11\\6\\\hline 5\end{array}$	$\begin{array}{r}12\\6\\\hline 6\end{array}$	$\begin{array}{r}13\\6\\\hline 7\end{array}$	$\begin{array}{r}14\\6\\\hline 8\end{array}$	$\begin{array}{r}15\\6\\\hline 9\end{array}$
$\begin{array}{r}8\\7\\\hline 1\end{array}$	$\begin{array}{r}9\\7\\\hline 2\end{array}$	$\begin{array}{r}10\\7\\\hline 3\end{array}$	$\begin{array}{r}11\\7\\\hline 4\end{array}$	$\begin{array}{r}12\\7\\\hline 5\end{array}$	$\begin{array}{r}13\\7\\\hline 6\end{array}$	$\begin{array}{r}14\\7\\\hline 7\end{array}$	$\begin{array}{r}15\\7\\\hline 8\end{array}$	$\begin{array}{r}16\\7\\\hline 9\end{array}$
$\begin{array}{r}9\\8\\\hline 1\end{array}$	$\begin{array}{r}10\\8\\\hline 2\end{array}$	$\begin{array}{r}11\\8\\\hline 3\end{array}$	$\begin{array}{r}12\\8\\\hline 4\end{array}$	$\begin{array}{r}13\\8\\\hline 5\end{array}$	$\begin{array}{r}14\\8\\\hline 6\end{array}$	$\begin{array}{r}15\\8\\\hline 7\end{array}$	$\begin{array}{r}16\\8\\\hline 8\end{array}$	$\begin{array}{r}17\\8\\\hline 9\end{array}$
$\begin{array}{r}10\\9\\\hline 1\end{array}$	$\begin{array}{r}11\\9\\\hline 2\end{array}$	$\begin{array}{r}12\\9\\\hline 3\end{array}$	$\begin{array}{r}13\\9\\\hline 4\end{array}$	$\begin{array}{r}14\\9\\\hline 5\end{array}$	$\begin{array}{r}15\\9\\\hline 6\end{array}$	$\begin{array}{r}16\\9\\\hline 7\end{array}$	$\begin{array}{r}17\\9\\\hline 8\end{array}$	$\begin{array}{r}18\\9\\\hline 9\end{array}$

concretely by the use of actual instructional materials the following combinations

$$\begin{array}{r} 0 \\ +\,0 \\ \hline 0 \end{array} \qquad \begin{array}{r} 0 \\ +\,1 \\ \hline 1 \end{array} \qquad \begin{array}{r} 1 \\ +\,0 \\ \hline 1 \end{array} \quad \text{etc.''}$$

Perhaps a great many people think of zero as meaning nothing because of the fact that when it is used it signifies that there are no units, tens, hundreds, or whatever place the zero occupies in the number. Note that this use of zero involves numbers of two or more places which do express quantities. The zero is a place-holder in these numbers. As was pointed out in Chapter 2, the place-holder aspect is the important role of zero. How one can develop concretely and through actual instructional materials the concept that "nothing" plus "nothing" equals "nothing" is not easily seen.

The use of zero to indicate failure to score in games is a situation frequently employed in illustrating the zero addition and subtraction facts. Critical examination of such illustrations will reveal that even here the use of zero is more a case of showing that the child or team has had a turn than it is a case of showing that zero means nothing.

As has already been indicated in the preceding discussion, the first use of zero in addition and subtraction in life occurs when two-place numbers are encountered. Therefore, the teaching of zero in addition and subtraction should be delayed until that phase of arithmetic is undertaken. After a use for the zero facts has been encountered and the pupil sees a need for knowing the facts, systematic attention should be given to mastering the facts. Unlike the procedures recommended for mastery of the basic addition and subtraction facts the procedures recommended for mastering the zero facts place great reliance on generalizations. The two generalizations, that zero added to any number equals the number and that zero subtracted from any number equals the number, may be stated in various ways. Some teachers prefer to emphasize the fact that

zero is not a number but a place-holder and therefore when it is encountered in an addition situation there is no addition and the one number is just recorded. This is the procedure that many adults use.

In case the child has difficulty in grasping the generalizations regarding zero in addition and subtraction, systematic study of the separate zero facts such as $5 + 0 = 5$ and $8 - 0 = 8$ may be recommended. Such emphasis on individual facts should, however, be a sort of last resort.

As the basic addition and subtraction facts are developed from problem situations, the children should begin using the conventional ways of expressing these basic facts; that is,

$$\begin{array}{r}4\\+2\\\hline 6\end{array},\quad 4 + 2 = 6,\quad \begin{array}{r}4\\-2\\\hline 2\end{array},$$

and $4 - 2 = 2$ should have been selected as the most efficient ways to write or record the thought-processes. The use of *and*, *take away*, *less*, and other words that mean *plus* and *minus*, need not be entirely abandoned because the conventional expressions have been introduced. Here, as in other phases of instruction, a situation must be created in which the children will see a reason for using the conventional means of stating the facts. Perhaps direct consideration of different ways of writing the facts that the children have learned and demonstrated is one of the best procedures for showing the advantages of conventional ways. After all the children's ways of expressing a fact have been written on the board, consideration of the best method should be undertaken.

Intensive Study of the Basic Addition and Subtraction Facts

When the teacher believes that the majority of the class understand the processes of addition and subtraction and know how to prove any fact, a test on some of the facts should be given. This test may be a sampling from all the facts or it may include

only a sampling from the easy addition and subtraction facts. Not all the facts are used because the time required for such a test would be too long. The following instructions indicate the procedure to be used in testing. "Today you are going to take a test on some of the addition and subtraction facts you have been using. Write each answer as quickly as you can. As soon as you have finished your work, write on your paper the last number your teacher has written on the board. Then sit quietly while others finish."

As the children take the test, the teacher keeps time. At the end of each minute the teacher writes on the board the number which indicates the minute just ended. When all or nearly all the children have finished, the teacher calls time.

The procedure may vary from this point. Either the papers may be collected to be scored at a later time by the teacher or they may be scored immediately by the pupils as the teacher gives the correct answers. After the papers are scored, the scores and the time required by some of the pupils should be written on the board. Those children who made a high score in a relatively short time should be asked to tell how they got the answers. The same question should be asked of those who either scored low or took a relatively long time to finish. The children who made high scores quickly will invariably state that they "just knew" the answers. Those who worked slowly will usually describe some method like putting down marks, breaking the numbers into parts, and making several operations out of one. The more cumbersome procedures should not be condemned, but the teacher should use this opportunity to call attention to the loss of time that results from the use of such methods. The need for knowing the facts automatically will be made clear through this emphasis on the difference in time.

The problem of how to learn the facts should next be raised with the class. The children should be permitted to make sug-

gestions before the teacher offers a learning or study procedure. The suggestions of the children should be written on the board. The following is typical of lists that children offer in addition:

1. Make marks to show how many and then memorize that.

$$4 + 3 = 7$$

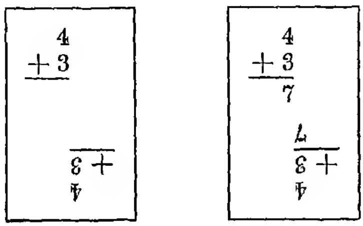
$$\mathrm{IIII} + \mathrm{III} = \mathrm{IIIIIII}$$

2. Write all the facts.
3. Hold your finger over the answer and try to say it
4. Say "4 and 3 are 7" over and over.
5. Play games like twinks.
6. Play guessing numbers with someone.

The teacher should call attention to the value of each of these suggestions, or, if a certain procedure is poor, point out why it should not be used. Letting children themselves offer study techniques has several purposes. It centers the attention of the child on how to study, the major thing that he will be doing for the next ten or fourteen years. It also gives the child confidence and some basis for evaluating the study procedure that the teacher or the textbook offers.

After emphasizing the value of the children's suggestions, the teacher can introduce one of the most important study techniques in the following way. "I have a method of study that some of you may like. I'll explain it. Each of these cards has a combination on it. On one side only the combination is given. [See illustration *a*.] You are to give the answer. The other side has the combination and the answer. [See illustration *b*.] One of the best ways to study with these cards is to take a pack with the 'combinations only' side facing you. Try to give the answer to the combination. If you can, put it aside in your 'known stack.' If you cannot or are not sure, turn the card over. Look at the answer; say the whole fact; then close your eyes and try to see the whole fact; say the whole fact — 'Four and three are seven' or 'Four plus three equal seven.' Next, open your eyes and see if you were right. If not, start

over. Continue until you can see the combination and the answer with your eyes closed. Then lay the card aside in your 'doubtful or unknown stack.' When you have gone through all the cards and located the ones you are not sure about, begin to study them, following the plan I've given you.



(a) (b)

"For the next few days we are going to take time for you to study the addition and subtraction combinations. You may follow any of the suggestions we have made. I believe it would be a good plan for you to show first by drawings that you can find the answers to items you missed on the test."

Permitting the children to select their own study procedures is done to foster interest and a good work spirit. It is admitted that some children will use the less efficient methods of study (by adult standards), such as games or writing the combinations. If, however, the use of these procedures makes for a better attitude toward work, the loss of effort will be justified. From what has been stated, it should be clear to the reader that the flash-card method suggested by the teacher, or a similar method of study of a list of examples in a book, on a sheet of paper, or on the blackboard, would be the best way to learn the basic addition and subtraction facts. Several approaches to the job of mastering a basic fact produce more interest in learning and therefore a more thorough learning of the fact than one method can, however economical of time and effort that method may be. Con-

sequently, a teacher is justified in tolerating some methods that may seem inefficient. The maintenance of interest is so important that the teacher may even suggest some of the less efficient means of study that children like to engage in. One of the most popular of these inefficient learning procedures is that of letting one child present the flash cards while another tries to give the answer to each combination. In this procedure the child presenting the cards learns very little.

From these remarks it should be evident that the writer thinks no hard-and-fast rules can be applied to a class that is learning the basic facts for automatic mastery. The main things for the teacher to be concerned about are: (1) that the children understand what they are doing; (2) that they know why they are doing it, and (3) that they have at least one systematic method of study. In studying the basic subtraction facts the question of how to "say the whole combination" arises. Consider, for example, $8 - 5$. Should the child say, "Eight minus five," "Eight less five," "Eight take away five," "Five from eight," "Five and what number equal eight," or some other statement? The first four of the five statements given indicate a take-away method of subtraction. The last one indicates an additive method. Since the take-away method is the one in most common use, and is the one most likely to be used by children, that is the method recommended. Again, on the basis of frequency of use, the statement "Five from eight" is recommended.

In the description of the systematic study procedures given above, both addition and subtraction facts were to be studied simultaneously. Many teachers, however, prefer to work first on addition and then on subtraction during the period when the facts are being studied for automatic mastery. This separation during the intensive-study period and a simultaneous presentation of the two processes during the developmental period is not inconsistent. Since the intensive-study period is a time for

fixing facts, little is to be gained in the way of providing relationships by studying the two processes simultaneously.[1]

If children have done the many number-concept-building exercises and the problem-demonstrating exercises in which they should engage before beginning the systematic study of number combinations, the teacher is justified in feeling reasonably sure that they understand the facts which they will try to master. However, because a class must move along without waiting for every pupil to reach a specified point of knowledge and experience, it is almost certain that in every class there will be some pupils who have not achieved as thorough an understanding of the basic facts as they need before practicing to master them. As a precautionary measure, therefore, the teacher should frequently require all children, as they study, to demonstrate with objects or marks the truth of the facts they are stating with numbers. During class periods of oral work, a similar use of objects or other means of demonstrating the truth of answers will also promote and check pupils' understanding of the facts.

If the time tests recommended earlier have been used adequately and the results discussed thoroughly, the pupils will probably have established in their own minds good reasons for acquiring automatic mastery of the basic facts.

To reinforce the necessity, however, the teacher may give one-item oral tests in which each pupil raises his hand as soon as he knows the answer to the question or in which he writes down the answer as soon as he has it. In work of this type, those pupils who do not know the facts will require more time for their indirect solutions and will, therefore, appear at a disadvantage. The teacher can then motivate this situation by emphasizing

[1] The reader should note that in the presentation of the teaching of addition and subtraction nearly all the examples used were from addition. The two processes, however, are so nearly alike that to repeat the corresponding subtraction teaching situation would be practically a duplication of what has already been said.

how easily the slow worker can overcome the disadvantage through learning the basic facts so that he can give them with promptness and accuracy.

Often children make slow progress in learning the basic facts because they are not following any systematic learning procedure. Children who either have not learned good methods of study or have not been convinced of the superiority of such methods will be found even in classes in which study procedures have been discussed and adequately demonstrated. In order to ascertain the methods of study used by those who are having difficulty in learning the number facts, the teacher should ask them to study aloud or to tell how they study. Even these means will not always reveal to the teacher the method or lack of method that the child follows. Children are frequently aware of good procedures and under test conditions will use them. For example, one child when asked to study aloud gave an illustration of the best study procedure, but as soon as he resumed unobserved study, he omitted the important steps of closing his eyes, saying and trying to visualize the whole fact. This child devoted the same amount of time to every combination he was studying. That he knew all combinations to the same degree of learning does not seem probable. The teacher, therefore, must be very much on the alert when making an inquiry into methods of study that a child uses.

The child's recognition of the values of undivided and intense effort in studying facts should be one of the goals of teaching. Like so much of the work of the teacher, this end is not accomplished by admonition alone or by any other single procedure. It is reached through constant and careful attention to the many opportunities for guidance which occur in the everyday teaching situation and by creating situations which will give children a chance to have the satisfaction that comes when a task is thoroughly done.

Because drill or practice is often considered to be in disrepute,

special attention is called to the emphasis given to practice or drill procedures recommended to fix the facts during the intensive-study period. Drill on the basic facts is necessary if the child is to have that mastery which is essential for good work in arithmetic. If he is to work efficiently in the subject, the handling of these frequently used facts should require little thought. The drill or intensive-learning period is then a means of meeting the child's needs. Automatic mastery of the facts is desirable, and intensive drill or practice is one of the most economical ways of acquiring such mastery. For further discussion of the place of drill in the teaching procedure, see Chapter 14.

To help motivate this intensive-study period, the teacher may suggest that as soon as children are sure they know all the facts, they may time themselves on one or several tests. A few moments of rapid group flash-card testing may also be used. In such a test the teacher presents flash cards before the entire group of pupils so rapidly that only those who have real mastery of the facts can give the correct responses. Usually the cards are taken from the front of the stack, although the placing of a new card in front of the last one presented is equally effective. The teacher should recognize that the use of flash cards with the group is only for motivation. There is no point in having the whole class drill on the same combinations. Flash cards then have little value as a class or group teaching device.

During the intensive-study period many uses can be made of lists of the basic addition and subtraction combinations. With such lists the procedure described below has been used to advantage.

"Here are the 81 basic addition examples for which you must know the answers if you are to add well. Try to give the answer to each example as soon as you look at it. If you cannot give the answer immediately, mark that example with your finger and then copy it on a piece of paper."

After the child has tried each example as directed above, he

is told to study the combinations he has written on his paper by one of the methods described on page 135.

A second procedure involving a list of the combinations is given in the following directions· "A good way to find out whether you really know the facts is to tap your desk or table with your finger as you give the answer to each combination. If you know the facts your tapping will be regular. If you do not know all your facts your tapping will not be regular." The children are then directed to study combinations they do not know by one of the methods already described. In this procedure a pupil may give wrong responses without in any way interfering with his tapping rate. However, it should be recalled that this study procedure is only one of many.

Although not the best type of material to use for intensive study, some problems introduced at the right time can be very valuable in that work. Problems can be used to bring the class back together (see lesson 4 for third grade, Chapter 14). They may also be used to provide a setting for number statements which are to be proved.

In learning those basic facts of addition in which the sum is greater than ten, and in subtractions in which the minuend is more than ten, children who have learned to regroup numbers into tens and ones will often resort to complementary methods of adding and subtracting the numbers involved. For example, instead of thinking directly the answer to 8 and 5, they regroup it as 8 and 2 and 3. And instead of thinking directly 13 minus 5 equals 8, they regroup the numbers as 10 minus 5 plus 3. Whether to discourage this roundabout procedure is a question on which teachers disagree. The writer believes it is a distinct advantage to let children use this method, especially in the second and third grades. Pupils who use such a procedure understand it. Speed at this level is a factor of only minor importance. The pupils' understanding of the procedure and the freedom from doubt about the results of its use are major

aims at this stage of learning. In later school work, the basic facts will be used so frequently that the direct method of thinking the answer, without any intervening step, will become automatic.[1]

Teaching Addition and Subtraction Together

In the instructional procedures described, addition and subtraction have been dealt with simultaneously. As was pointed out earlier, this is believed to be an advantage. Critics of the plan contend that confusion will result. They contend that the child confronted with $4 - 3 = 1$ before he has learned $4 + 3 = 7$ is likely to use 7 and 1 interchangeably. If the use of abstract number symbols is undertaken before the child has had adequate experience in counting or with the concept-building exercises, confusion may result from teaching the two processes together, but if a meaningful program has preceded systematic learning of the facts, there need be no fear of confusion. The child who understands 4 knows not only such facts as that it is made up of 2 and 2; 3 and 1; 2, 1, and 1; but also that 4 is 2 less than 6, 3 less than 7, and so on. Thus teaching the two processes together makes for better understanding.

Order of Teaching

It should be noted that no special order of difficulty of the facts was used in the recommended plan of instruction. The facts involving small numbers (the easy addition and easy subtraction facts) should be the first concern of every beginner. This recommendation is in harmony with the general belief that initial work should deal with the simpler aspects of a process and it is also in harmony with one of the major characteristics of our number system: namely, that the first numbers are dealt with as ones. Furthermore, the primary emphasis in counting and in the concept-building program was on the ones numbers.

[1] For a similar point of view see H G Wheat, *The Psychology and Teaching of Arithmetic* (Boston: D. C. Heath and Company, 1937), p. 296.

Aside from this division of the facts, no attempt is made to present facts in any order. Children should have many experiences with a fact before they are asked to learn the abstract statement of that fact. If these fundamental number experiences are derived from lifelike situations, they will not involve first the easy and then the more difficult facts. Furthermore, since every basic fact must be learned, the relationships that will be recognized as children work with all facts will be beneficial in getting an understanding of a specific fact. Thus, order of difficulty cannot be a major determinant in teaching the basic addition and subtraction facts for mastery. It should be recognized that the doubles in addition are usually easy and that children then use these facts in determining other addition facts. For example, 8 + 9 is recognized as being one more than 8 + 8. As was recommended in the case of the complementary method of addition and subtraction, children should be permitted to use an indirect method of this type.

It should also be noted that addition and subtraction work up to this point has not been limited to the basic facts. Two-place numbers, as in the oral addition of one ten and two tens to a number on the number chart, are used in the latter part of the first grade and in the early part of the second grade. While the major work is on the basic facts, the development of the idea that tens can be handled in the same manner as ones is very important. Then, too, this addition of tens is important in developing an understanding of numeration and notation.

The recommendations concerning the order of teaching are not made without consideration of the findings of the many studies [1] which have attempted to ascertain the order in which

[1] Frank L. Clapp, *The Number Combinations. Their Relative Difficulty and the Figuring of Their Appearance in Textbooks* (Madison, Wisconsin· University of Wisconsin, 1924), F. B Knight and Minnie Behrens, *The Learning of the 100 Addition Combinations* (New York. Longmans, Green and Company, 1928), H. V Holloway, *An Experimental Study to Determine the Relative Difficulty of the Elementary Number Combinations in Addition and Subtraction* (University of Pennsylvania, 1914)

facts should be learned. These studies have been examined, but their value has been rejected on the grounds that they were based on learning procedures very different from those recommended by the author of this book. For example, none of the studies was concerned with developing the idea that tens can be added just as ones are. Furthermore, the studies give no indication of the practice of stressing relations between facts. During study of facts for automatic mastery, the lists of addition or subtraction facts are often divided, because the entire lists are too long for convenient instructional or learning procedures. In making the division, the easy facts are taken up first. If further division is necessary, a random selection of facts for each lesson or assignment is recommended.

Summary of the Methods Used

Now that the procedures to be used in learning the basic addition and subtraction facts have been given, it seems appropriate to summarize the major features of the method of instruction used. Since an adequate foundation for addition and subtraction is a basic factor of the method of instruction, the summary includes a review of that foundation.

1. The foundation for addition and subtraction consists of the following important items: Children who are to be taught the addition and subtraction facts are given the benefit of the concept-building program, including an adequate counting experience. In counting, they are given a procedure by which they can solve any addition and subtraction situation. They are also given much experience with different ways (objects, marks, dots, drawings, pictures, words, and symbols) of representing number. In addition, the children have experience in putting groups of objects together into one group and taking them apart to make new groups. Through the use of objects, they solve mentally many addition and subtraction problems.

2. Following the foundation program, systematic instruction is begun in the following manner: First, the children are presented with problem situations which they solve by methods already known. Second, they then make records of the thought-processes by which they arrive at the solutions. These records of thinking are evaluated and one (e.g., $\begin{array}{r} 4 \\ +3 \\ \hline 7 \end{array}$ or $4 + 3 = 7$) is selected as the best statement of the fact illustrated. The pupils also learn, as a result of this demonstration of the problem situation, the nature of each process and the reason why the process is of value; that is, they learn that addition and subtraction are a way of regrouping quantities and that this grouping makes it easier to think further about the quantities involved. Third, through the timed test situations, the children are shown the advantage of learning the facts so thoroughly that each fact can be given automatically when the combination is presented. Fourth, in addition to being made aware of a reason for mastering facts, the children are confronted with the problem of finding methods for intensive study of the facts. Although permitted to use their own methods of study, they are also acquainted with the most efficient methods of study known. Throughout the intensive-study period, the children are frequently asked to demonstrate their understanding of the facts they are trying to master.

3. A summary of the recommended method used in teaching the basic addition and subtraction facts would be incomplete without considering, first, how this method differs from those generally used, and second, how it may be adapted to or made a part of the generally used methods. The outstanding difference between the method recommended and those in common practice is in the initial systematic instructional procedures. Instead of considering that it is the function of problems to create situations out of which a need for more efficient ways of

handling such situations may arise, the conventional methods utilize problems to illustrate the addition or subtraction facts, and then provide various procedures for learning the facts so illustrated. The other differences are not so marked. Conventional methods do not include the use of proof, emphasis on the nature of the processes, or children's suggestions and use of various ways of study. With slight changes in or additions to textbook procedure and with corresponding explanations and suggestions in the teachers' manuals, these last three features of the recommended method might be easily incorporated with commonly used methods.

Addition and Subtraction with Two-Place Numbers

The child's first experience with the addition of two-place numbers occurs in the first grade in connection with study of the number chart. For example, children are asked to find the number that is two tens more than 30. In oral exercises early in the second grade, children are asked to add such numbers as 20 and 30, 22 and 40, 26 and 31. These exercises are concerned primarily with emphasizing the idea that tens are a collection and that they can be handled just like ones. Nevertheless, the procedure does involve the addition of two-place numbers. Systematic instruction in addition and subtraction of two-place numbers does not begin, however, until the children have had systematic instruction in the basic facts.

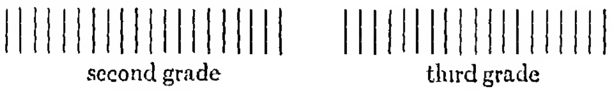
This new work is introduced by presenting the children with four or five problems that contain quantities requiring the use of two-place numbers. The first problems should involve the addition and subtraction of even tens. By even tens is meant such examples as $30 + 20$, $20 + 40$, $40 - 20$, and $60 - 30$. The children are asked to write their answers to the problems. They are then told to show by diagram, by the use of tens blocks or other means, that their answers are correct. Finally, they

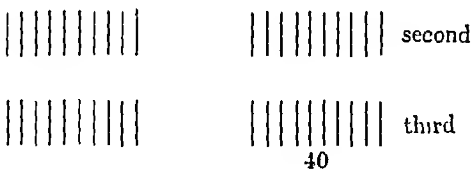
are asked to show with numbers what their diagrams show. In other words, a number record of their thinking is required.

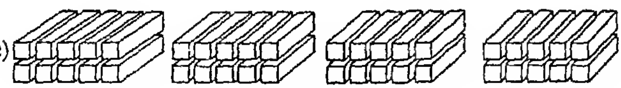
The procedure is well illustrated by the following methods which one class used to record how they thought out the answer to a problem

Problem: At the party last week, there were 20 children from the second grade and 20 from the third grade. How many children were there at the party?

Solution by using marks and blocks:

(a) 

(b) 

(c) 

(tens blocks) = 40

Solution by using numbers:

(*a*) $20 + 20 = 40$

(*b*)
$$\begin{array}{r} 20 \\ \underline{20} \\ 40 \end{array}$$

(*c*) 2 tens
<u>2 tens</u>
4 tens = 40

(*d*) $20 + 10 = 30$
$30 + 10 = 40$

After the solutions of several problems had been discussed by the class, the teacher gave the following assignment: "How does the addition and subtraction of tens differ from the addition and subtraction of ones?" The gist of children's answers to this question is summarized in the following. "It is the same except that you have to write a zero to show that it is tens you are working with."

In order to test the truth of this generalization, a number of examples involving the addition and subtraction of tens and the corresponding ones were then solved. After the children had worked sufficient examples to prove that the addition and subtraction of tens was very similar to the addition and subtraction of ones, the teacher directed attention toward the selection of the best method of writing such examples. Of course, type *b* of the number records was the method accepted.

Following these exercises, problems involving tens and ones (no borrowing or carrying) were introduced. The following are examples of this type: 21 + 42, 33 + 36, and 32 − 21. The records of the way the children thought through these examples were similar to those shown above, except that in type *c* the number of ones had to be written. Practice examples were then worked so that the children might have an opportunity to fix the process.

Carrying and Borrowing

This new phase of arithmetic should be introduced through problems just as was initial instruction in addition and subtraction with two-place numbers. Furthermore, the procedures are similar to those already used. Of course, the children will encounter more difficulties than they did where no carrying or borrowing was involved. Those who use the inefficient counting method (examples *a* and *b*, using marks, page 147) will not have much trouble in arriving at the answers, but will get little help from this method when they start to make their number records. Those who use tens blocks will probably also get the answers

without difficulty, especially if they have had much experience using the blocks to show the sum of numbers such as 7 and 4. Some of these children may even be able to explain their number records. However, most groups will need some help in discovering the nature of carrying. This help should not be given immediately. The children should struggle with the difficulty. In this way they will be prepared for the explanation. If the best record of thinking is not presented or understood by the class, the teacher should invite the whole class to share in the solution of one problem.

Suppose this problem involves $37 + 26$. Some children may be able to give the answer orally and may commonly explain their steps something like this: $30 + 20 = 50$; $50 + 13 = 63$. The procedures such children use in adding the numbers orally seldom help to explain carrying. They are asked, therefore, to show with tens blocks and ones blocks the quantities involved. Then the familiar procedure of doing what the problem requires, putting all the quantities together into a single group, should be followed. The simplest way of showing this quantity is the next issue to be raised. Of course, the substitution of 1 tens block for 10 ones blocks will simplify the situation. The teacher should then say, "Now, let's do that with numbers." The quantities are first written thus:

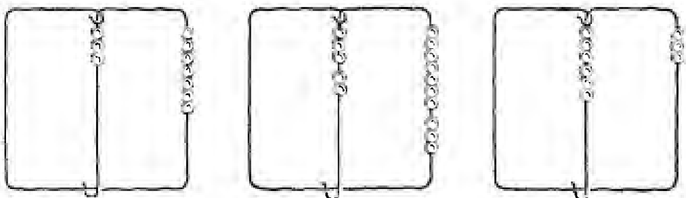
3 tens 7 ones
2 tens 6 ones
5 tens 13 ones (13 ones = 1 ten and 3 ones)
5 tens and 1 ten and 3 ones
6 tens and 3 ones = 63

After several problems have been worked in this way, the numbers should be written in the conventional manner and the addition performed as follows:

$$\begin{array}{r} 37 \\ \underline{26} \\ 63 \end{array}$$

7 and 6 equal 13, but we can write only ones in the ones place, so only the 3 is written. The one ten is written as a small number with the other tens and then added there.

The abacus is also a good device for teaching carrying. The following illustration shows how the abacus would be used in adding 37 and 26: First, 37 is registered on the abacus by sliding up the appropriate ones and tens beads (see figure *a*). Then an attempt is made to put 26 on the same abacus. Of course,



(a) (b) (c)

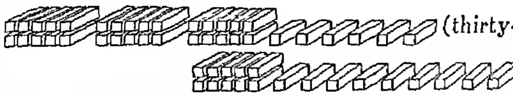
the 2 tens can be added without difficulty, but when the child attempts to register the 6 ones he finds only 3 ones beads that have not already been used. These 3 ones are used. The abacus then appears as in figure *b*. The ten ones can of course be represented by 1 bead in the tens row. This substitution or change is made and the 3 remaining ones of the 6 ones are then registered on the ones row. The abacus after completion of this operation is shown in figure *c*.

Borrowing is introduced and taught in a manner quite similar to that followed in carrying. Again chief reliance is placed on the explanation with the tens blocks and on writing the numbers as tens and ones.

The method is well illustrated in the description of the procedures followed by one third-grade class. Along with other problems the following was presented: "We set out 36 tomato plants, but now only 19 are alive. How many have died?"

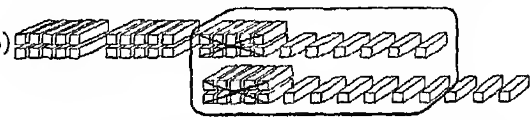
In the illustrations showing how this problem could be solved the usual device was employed of drawing marks and counting. Tens blocks were also used, and subtraction was done by tens.

Since the children had difficulty in finding a good number record of thought, the teacher suggested recording with numbers what was done with the tens blocks.[1] Here is the procedure with tens blocks.

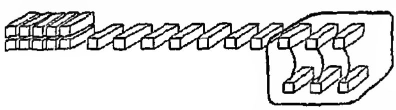
(a)  (thirty-six set out)

(nineteen still living)

In this step the children merely represented with blocks the number of plants set out and the number that remained.

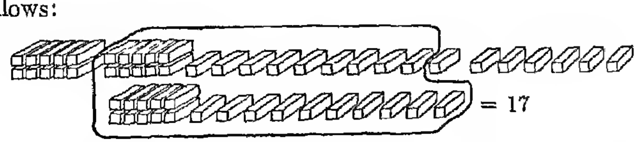
(b) 

In this second step the blocks representing the remaining plants were paired with blocks representing plants set out. Three of the ones blocks representing plants remaining were left unpaired.

(c)  = 17 died

For one of the tens blocks left unpaired in step *b* was substituted its equivalent of ten ones blocks and then the three blocks representing remaining plants were paired.

In this procedure the tens and ones of the subtrahend were paired with the tens and ones of the minuend. Since there were three more ones in the subtrahend than in the minuend, one of the tens of the minuend had to be changed to ones. In an effort to shorten the procedures, steps *b* and *c* were combined as follows:

 = 17

[1] Bundles of sticks (ten to a bundle) and individual sticks could be used instead of tens and ones blocks.

The number record of this procedure is shown here.

$$\begin{array}{lcl} 3 \text{ tens and } 6 \text{ ones} &=& 2 \text{ tens and } 16 \text{ ones} \\ 1 \text{ ten and } 9 \text{ ones} &=& \underline{1 \text{ ten and } 9 \text{ ones}} \\ && 1 \text{ ten and } 7 \text{ ones} = 17 \end{array}$$

Shown entirely with numbers, this record becomes

$$\begin{array}{r} \overset{2}{\cancel{3}}\,{}^{1}6 \\ \underline{1\;9} \\ 1\;7 \end{array}$$

Questions such as the following will help children to clarify their ideas of the all-number record. "What does the little one beside the 6 mean?" "Where does it come from?" "Why is a line drawn through the 3 and a little 2 written above it?"

In answering these questions frequent reference should be made to the blocks and to the intermediate number record. In so doing, it should be made clear that the little 1 is read with the 6 as sixteen; that it corresponds to the sixteen ones blocks; that marking out the 3 and writing a 2 is the same as removing one of the tens blocks and putting in its place ten ones blocks.

The topic of borrowing usually raises the question of the type of subtraction to be used. There are at present several different methods of subtraction, but all are essentially variations and combinations of a take-away or an addition plan. In solving a basic subtraction situation such as $8 - 3$, a pupil using the take-away method would reason somewhat as follows: What number remains when 3 is taken away from 8? A pupil using the addition method would reason somewhat as follows: What number added to 3 will result in 8?

In situations involving borrowing the two basic methods of subtraction require markedly different procedures and the variations within each method entail different steps. The solutions of the same example using one form of the take-away method and one form of the addition method illustrate the main differ-

ences. Consider, for example, the following subtraction situation:

$$\begin{array}{r} 36 \\ -\ 19 \\ \hline 17 \end{array}$$

In the take-away method the procedure would be explained as follows: Since 9 ones cannot be taken from 6 ones, a ten is borrowed (changed to ten ones) from the 3 tens, making 16 ones. Nine ones from 16 ones leaves 7 ones. The 7 is written in the ones place. One ten from the remaining 2 tens leaves 1 ten. The 1 ten is written in the tens place. This form of subtraction is commonly called the *take-away borrow method.*

In one form of the additive method the procedure would be explained as follows. Since no positive number can be added to 9 to equal 6, the 6 must be only part of the number 16. Nine ones and 7 ones equal 16. The 7 is written in the ones place. One ten (from subtrahend) and 1 ten carried from 16 equal 2. Two tens and 1 ten equal 3 tens. The 1 ten is written in the tens column. This form of subtraction is sometimes called the *complementary addition method.*

In the take-away borrow procedure the two number questions to be answered are: (1) "9 from 16 equals what number?" and (2) "1 from 2 equals what number?" In the addition procedure the two number questions are: (1) "9 and what number equal 16?" and "2 and what number equal 3?" Of course, those who are proficient in either method do not think such questions but think immediately "9 from 16, 7" or "9 and 7, 16."

A major issue in subtraction is concerned with how the minuend number is increased by ten. In the take-away borrow situation described, one of the 3 tens was changed to 10 ones leaving only 2 tens in the tens place of the minuend. In the addition situation it was assumed that the 6 in the minuend was only the ones part of 16. For both the take-away and the addition methods of subtraction there is another widely used

method of increasing the minuend numbers by ten. In brief, this other method consists of increasing the ones minuend number by adding a 10, and then, to balance this addition, adding a 10 to the tens number of the subtrahend. This is known as the *method of equal additions,* that is, equal amounts are added to both minuend and subtrahend. The procedure when the equal additions method is used in solving 36 − 19 (take-away subtraction) would be somewhat as follows: "9 cannot be subtracted from 6. Add a 10 to the 6 to make 16. To maintain the same conditions between minuend and subtrahend, add 10 to the 1 ten of the subtrahend. Then 2 tens are subtracted from the 3 tens." If the addition method of subtraction is used, the procedure would be somewhat as follows: "There is no number to be added to 9 that equals 6. Therefore, 10 is added to the 6. To balance this addition of 10 to the minuend a 10 is added to the tens figure of the subtrahend."

While this method of equal additions is mathematically sound, teachers of arithmetic should recognize that children who are just being confronted with borrowing in arithmetic have not had the algebra which will enable them to see the fundamental principle involved. The method of equal additions is therefore more difficult for children to understand than is the borrow method. To convince yourself of the truth of this statement, try using both methods in explaining to a child or to a class how 13 is secured as the answer to the following problem: "Of the 32 fifth-grade pupils, 19 are present. How many are absent?"

The purpose of addition is to find a total or the result when two or more numbers are combined into a single group. But in this case that total, 32, is already given. This 32 must be divided into two groups, one of which is known (the 19 present). The logical thing to do is to identify the 19 and then find how many of the 32 are left. In other words, 19 of the 32 are removed — taken away. Removal or taking-away, either by actually moving objects or by marking, or by identifying in some other

way, is the process children will use if asked to show how such a problem can be solved. Notice the illustrations for the sample problem on page 156. To show the steps in the additive method, either by means of marks or objects, without taking away, is very difficult. Two rows of marks, each containing 32, would be required. The reasoning would be something like the following: "Here are 32 marks representing the children of the fifth grade. These 19 (in row below the first) show how many children are present. These others (remaining marks in second row) show how many children are absent because the marks show how many must be added to 19 to make it equal to 32 (the total in the first line)."

A great many proponents of the additive method contend that it should be taught because it is the method used in making change. Making change, however, is not a process of subtraction at all, but merely of counting. Consider, for example, the procedure when you purchase a 20-cent article and hand the clerk one dollar. If there is a one-cent sales tax, the clerk hands you your purchase, saying, "twenty-one, twenty-two, twenty-three, twenty-four, twenty-five, fifty, one dollar." At no time did the clerk subtract 21 from 100. Instead of being evidence in favor of the additive method of subtraction, the method of making change is good evidence for the counting method. Since "take-away" is the method that most of us use and because it is more logical, that method is recommended. For a detailed discussion of methods of subtraction, the reader is referred to Morton.[1] Other references that deal with certain phases of this topic will be found at the end of this chapter. Special attention is called to the work of E. A. G. Lamborn and of J T. Johnson, and to the report of the Joint Committee of the American Educational Research Association and the Department of Classroom Teachers of the National Education Association.

[1] R L Morton, *Teaching Arithmetic in the Elementary School* (New York: Silver-Burdett Company, 1937–38), I, 176–88

Borrowing is such a difficult phase of subtraction to teach that more detailed consideration of the topic seems worth while. To give discussion of the topic a practical setting, an illustrative problem and a children's solution of that problem are presented.

Problem. Of the 32 fifth-grade pupils 19 are present. How many are absent?

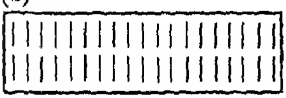
Examples of illustration:

(a)

13 absent

+++++++++++++++++++ ||||||||||||| $32-19=13$

(b)



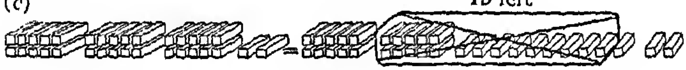
13

|||||||||||||

$$\begin{array}{r} 1 \\ 2\cancel{3}2 \\ \underline{19} \\ 13 \end{array}$$

(c)

13 left



2 tens	12 ones
− 1 ten	9 ones
1 ten	3 ones

Examples of number record:

(*a*) $32 - 19 = 13$

(*b*) $$\begin{array}{r} {}^{2}\cancel{3}{}^{1}2 \\ \underline{1\ 9} \\ 1\ 3 \end{array}$$

(*c*)

	2 tens	12 ones
−	1 ten	9 ones
	1 ten	3 ones

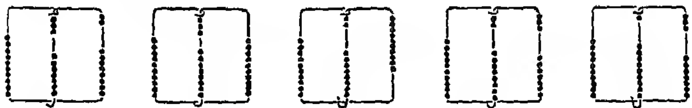
The explanation for the example of illustration *c* was something like this: First, the 32 was represented by 3 tens blocks and 2 ones blocks. Then the 19 present were represented by the 1 tens block and 9 ones blocks. It was clear that there were not enough ones to match the nine ones to be removed. In

order to supply the necessary ones, ten ones blocks were substituted for one of the three tens blocks representing the thirty-two. The rest of the operation is obvious. In the number-record explanation *b*, the term *borrow* does not have to be used Instead, the pupil can say one of the tens in the 3 is changed into ones. To show this, we write a small 1 in front of the 2 in the ones column. To show the remaining 2 tens, we cross out the 3 and write a 2 above it. *Change* instead of *borrow* would be a much better word to use with beginners. But because the term *borrow* is so commonly used today, it may be wise to retain it even though it is a misnomer.

The writing of the borrowed 1 is often strenuously opposed by teachers of arithmetic on the ground that it is a useless crutch. From a logical and practical point of view, however, it seems to be an aid to understanding and its use should be permitted. It relieves the mind of the burden of remembering that one ten has been removed from the tens and it is certainly easier to subtract from a visible than from an invisible number The writing of the borrowed 1 is no more a crutch than is the long form of division in problems with one-figure divisors. Since few people now oppose the teaching of the long form of division before the short form, one may safely assume that the writing of the borrowed 1 will be accepted by teachers who are genuinely concerned about teaching arithmetic in a meaningful way. Of course, children should eventually acquire enough confidence in their ability so that they can abandon the writing of the borrowed 1. However, many an adult writes the borrowed figure when making a subtraction that is crucial; and the practice should not be denied children.

To show the meaning of borrowing as well as carrying, teachers will find the abacus a helpful device. The five figures which follow illustrate how 19 is subtracted from 32. Figure *a* merely shows the 32 registered on the abacus. In figure *b*, two of the 9 ones have been subtracted, and in figure *c* one of the 3 tens is

changed to 10 ones. The remaining 7 ones are then subtracted by dropping 7 of the 10 ones, as in *d*. The one ten of the 19 is then subtracted by dropping one of the tens beads. The completed process is shown in *e*.



(a) (b) (c) (d) (e)

When the processes of carrying and borrowing are understood, the children should practice with examples. The textbook is probably the best source of such practice examples. Good sets of examples can also be constructed by teachers or supervisors. The children should be required to show during the intensive-study period that they understand the process they are attempting to fix.

Column Addition

Column addition is taught before addition and subtraction with two-place numbers receives much attention. In fact, the beginning of column addition is found in the exercises requiring the child to put the proper number of people in three or four houses, or to think of three numbers that make up one number such as 8. There are also many life problems that arise in the first and second grades which require column addition.

Systematic instruction should follow the same lines as those recommended for the basic addition facts. Objects, marks, and the making of tens should all be employed. The work in the second grade, except in the special case of class projects, should not involve more than four or five numbers.

Examples of only four or five numbers will involve *higher decade addition* (the adding of ones to tens; e.g., 17 + 2, 24 + 2, etc.). Many texts and courses of study require children to learn these higher decade addition facts. If some attention

needs to be given to a higher decade addition fact, the relation between that fact and the basic fact should be emphasized. For example, 17 + 2 is connected with 7 + 2. Teachers who wish to have their pupils work with higher decade addition facts should give special attention to such of those facts as are used for carrying in multiplication. For example, when a pupil is using 6 as a multiplier and meets 6 × 4 in a problem, he may have to add to the product 24 a 1, 2, 3, 4, or 5, and add the carried 2 to 12, 18, 24, 30, 36, 42, 48, or 54 In multiplying, the pupil needs to know how to add quickly and accurately certain carried numbers to 12, 14, 15, 16, 18, 21, 24, 25, 27, 28, 30, 32, 35, 36, 40, 42, 45, 48, 49, 54, 56, 63, 64, 72, 81, but not to other higher decades such as 23, 26, 29, which are not products used in basic facts. Higher decade addition examples are commonly classified into two groups, those without bridging (e g., 27 + 2 = 29 and 32 + 6 = 38) and those with bridging (e.g., 27 + 6 = 33 and 46 + 8 = 54). Bridging occurs when the sum is in the decade higher than the larger addend.

In column addition the chief difficulty arises through the fact that one of the numbers added after the first addition is not visible. Unfortunately, no satisfactory way to avoid this difficulty has yet been found. Emphasis on oral work may be one procedure that will help children to master the difficulty. However, there is no evidence to show that oral work makes for greater skill in column addition.

A question which frequently arises in the teaching of column addition is whether or not children should be taught to group numbers. For example, when a child is confronted with 3 + 6 + 7 + 2 + 3 + 5, should he make 10 out of the 7 and 3 and then add the 6? This practice has not proved very helpful and usually makes for errors. Ideally, in adding a column like the above, the children should say 9, 16, 18, and so on, not 3 and 6 are 9, 9 and 7 are 16. As in other phases of arithmetic, there are many steps through which a child may pass before the

best procedure is accepted. In adding long columns (upper grades), children sometimes exceed their attention span. They should be taught to place a finger or a pencil on the last number added, keep repeating the sum, and glance away for a moment.

In column addition the question of whether to add up or down is usually raised. The question is really of minor importance, since children should establish early the habit of adding first down or up and then checking by adding in the opposite direction. Which way to add first appears to be of little consequence.[1] Those who favor adding down seem to have the best arguments on their side when it is considered that downward addition moves in the direction that we read, and if a pencil is used to hold the place the hand is in position for writing the sum. On the other hand those who add up insist that the pencil hand is at the bottom of the column when the last number is written and therefore it is natural to start adding up. Furthermore, there are good computers who add up on some columns and down on others.

Oral serial addition of the same one-figure number has been used with marked success to improve column addition. The following example will illustrate the procedure: In an oral exercise the teacher may say: "Begin with 9 and add sevens." The pupil responding correctly will say, "Sixteen, twenty-three, thirty, thirty-seven, forty-four," and so on. Note the various bridging that occurs in the above example. The numbers 5, 6, 7, 8, and 9 can be profitably used in such exercises, and by using various starting numbers (all teens except the double of the number to be added) a great many different situations are available.

[1] *The Implication of Research for the Classroom Teacher*, Joint Yearbook of the Department of Classroom Teachers and the American Educational Research Association of the National Education Association (1939), p 194.

Use of Cues in Teaching Addition and Subtraction Problems

Many arithmetic books make extensive use of cues or key words to help the child to decide which process to use in solving problems. For example, children are told that addition is used to find the sum, the total, how many altogether, and how many in all, and subtraction is used to find how many more, how many less, how many left, and how many remain.[1] The author questions the use of such cues. It takes the attention of the children away from the conditions stated in the problem by immediately directing their attention to the getting of the answer. While there is nothing wrong with getting an answer, that answer can be of little value if the child does not see the whole problem situation. In such instances answers merely enable the child to satisfy the artificial conditions set by text or teacher. Use of cues short circuits the essential analysis of problems. To illustrate, consider the following division cues: (*a*) "Find the average (given total)." (*b*) "$\frac{1}{4}$ square mile = . . . acres" (and the like). (*c*) "A articles cost B cents. One article will cost . . ."[2] Teaching pupils that division is to be used when such cues as those just cited appear in a problem is giving problem-solving greater emphasis than the author of this book advocates.

On the other hand, textbook writers may have resorted to the use of cues because the problem-solving ability of pupils is so notoriously weak. Some of the cues suggested, as for example that addition is used to tell how many in all, are partially incorrect. If problems are used to illustrate mathematical facts and procedures, the need for cues will be to some extent eliminated. Further discussion of this situation will be found under the topic of problem-solving in Chapter 7.

[1] For a book which recommends the teaching of cues see W. J. Osburn, *Corrective Arithmetic* (Boston: Houghton Mifflin Company, 1929), II, 169-71.

[2] W. J. Osburn, *op. cit.*, p. 227.

Other phases of addition, such as work with numbers of three or more digits, addition of addends of unequal length, and the like, do not confront the learner with any radically new procedures. The principles governing instruction as described in the preceding pages apply similarly to these new phases of arithmetic. Since it is difficult for the child to use the type of proof employed in beginning addition, he should be given other opportunities to evaluate and understand the procedures with which he is learning to work. To make the addition of large numbers significant, some room project which requires the use of such numbers should be introduced. A record, compiled as a part of a social studies unit, of the number of livestock received at some near-by packing center will help to make the addition of large numbers significant. Such a project has the further advantage of introducing the rounding of large numbers as a reasonable procedure. Other means of representing quantities, such as the pictograph and the bar graph, will help the child to grasp the idea that numbers can sometimes be rounded without a loss of essential information. In order to emphasize and give practice in the rounding of numbers, the addition and subtraction of large numbers should occasionally be used in the oral arithmetic period.

The subtraction of numbers of three or more places introduces one new difficulty: borrowing from zero, that is, when the figure from which borrowing would normally take place is a zero. The situation is illustrated in the following examples: 306 − 198 and 3000 − 1654. The long mental process of first changing one hundred from the three hundreds (see first example) to ten tens and then changing one of these ten tens to ten ones leaving only nine tens is perhaps essential in the first solutions of such problems, but eventually pupils should follow a procedure such as the following: "8 from 16, 8; 9 from 9, 0; 1 from 2, 1." Obviously, 6 can be made into 16 only by borrowing or changing a ten. Instead of thinking change one of the three hundreds

to ten tens it probably is best to think of the three hundreds as thirty tens. Then when one is changed, twenty-nine remain Some textbooks use money situations to introduce this phase of subtraction. For example, a problem involving the subtraction of $2.75 from $4.00 is presented and solved by changing one of the four dollars to nine dimes and ten cents and then removing or identifying the $2.75 and noting the remainder.

The student of the teaching of arithmetic will find it profitable to consult a number of third-grade arithmetic books on this phase of arithmetic. The actual teaching procedures presented in these textbooks will show the steps which the authors of the books intend for children's use. Regardless of how the procedure is presented and explained, subtraction when borrowing from zero is required is a difficult process for many children and therefore merits the careful consideration of teachers of arithmetic.

After the initial study of various phases of addition and subtraction, primarily in the third and fourth grades, the amount of practice required to maintain reasonable efficiency must be considered. No program of instruction has yet been developed in which the regular arithmetic work will be sufficient to maintain all pupils at a high level of efficiency in these two fundamental operations. Throughout the upper grades, then, some attention must be given to teaching addition and subtraction. The teacher should not feel apologetic about taking time for such work. The children should be led to see the need for the review or practice, should be provided with appropriate practice material, and should even be required to show that they understand the work on which they are practicing. The review exercises in texts and workbooks are good sources of such practice material. No attempt should be made to bring all children up to the same level of achievement, but some description of the child's achievement should become a part of his permanent school record. It is through such records that a teacher and

administrator get a good picture of a child's progress from year to year. The child should continue to progress in his ability to add and subtract as long as he studies arithmetic. Of course, if he becomes very proficient in the two processes, no further growth should be expected.

STUDY QUESTIONS

1 Three boys owned 16, 18, and 12 marbles respectively. Why is 46 a better statement of how many marbles the three boys had than is 16, 18, and 12? (*1*) Because 46 tells how many in all. (*2*) Because 46 is easier to visualize than 16, 18, and 12. (*3*) Because 46 is a tens and ones number. (*4*) N.

2. Why is $6 + 3 = 9$ a basic addition fact while $16 + 3 = 19$ is not? (*1*) Because the sum of the first is a one-digit number. (*2*) Because you can add 16 and 3 without knowing that $16 + 3 = 19$. (*3*) Because 16 and 3 are used only in column addition. (*4*) N.

3. Why is it logical to argue for teaching $6 + 1 = 7$ before teaching that $6 + 0 = 6$? (*1*) Because $6 + 1$ is needed in the addition of one-figure numbers. (*2*) Because $6 + 0$ is a zero fact. (*3*) Because $6 + 0$ is more difficult than $6 + 1$. (*4*) N.

4. Which of these is classified as an easy addition fact? (*1*) $6 + 6 = 12$. (*2*) $8 + 4 = 12$. (*3*) $9 + 3 = 12$. (*4*) N.

5. For what reason should a child be asked to drill or practice a basic subtraction fact such as $8 - 6 = 2$ if he already is able to figure out the fact and understands what he is doing? (*1*) There is no reason. (*2*) To save thought. (*3*) To make his answer more accurate. (*4*) N.

6. If you do not include the zero facts, how many basic addition facts are there? (*1*) 36. (*2*) 45. (*3*) 81. (*4*) N

7. What is the most serious disadvantage of the flash-card method of studying the basic addition facts? (*1*) It is an individual method. (*2*) It is difficult to motivate. (*3*) A wrong answer appears just as right to a child as does the right answer. (*4*) N.

8. For teaching which of these phases of addition are tens blocks most useful? (*1*) The hard facts. (*2*) Addition of tens and ones — no bridging. (*3*) Addition of two-place numbers with carrying. (*4*) N.

9. What makes the use of counting especially desirable in introductory addition work? (*1*) Counting is more closely related to addition than to any other process. (*2*) Children will have just finished their study of counting and will therefore not have forgotten how to count. (*3*) Children understand and have confidence in their ability to count. (*4*) N.

10. What is the major argument against the simultaneous teaching of addition and subtraction (basic facts)? (*1*) There are too many facts to learn. (*2*) Addition is the foundation for subtraction. (*3*) The process of subtraction is more difficult and should therefore follow addition.

11. Is it considered desirable practice to ask a child who is studying the basic facts for automatic mastery to prove a fact? (*1*) Yes. (*2*) No.

12. The complementary method of subtraction has one distinct advantage. What is it? (*1*) It is a rapid method. (*2*) It makes for accuracy. (*3*) Only the easy basic facts have to be mastered. (*4*) N.

13. The order of presenting the basic addition facts is not considered very important by the author of this book. What is one of the chief reasons for this lack of emphasis on order? (*1*) Recent research has shown that order has no great value. (*2*) Experiences with many facts before work for mastery emphasize relationships. (*3*) $9 + 6 = 15$ needs more attention than $6 + 2 = 8$, and therefore should be presented early. (*4*) N.

14. What is the difference between an addition combination and an addition fact? (*1*) There is none. (*2*) Facts involve two-figure numbers while combinations involve only one-figure numbers. (*3*) Facts do not include answers while the combinations do. (*4*) N.

15. What is the purpose of giving the timed test at the beginning of the period for intensive study of the basic addition and subtraction facts? *(1)* To give the child and teacher a benchmark from which to measure progress. *(2)* To set a goal for the child to work for. *(3)* To provide evidence which will help the child to see that he needs to master the facts. *(4)* N.

16. Since there are only very rare occasions when all children in a class will need to drill on the same fact, how can even a few minutes of flash-card drill with the whole class be justified? *(1)* It cannot be justified. *(2)* For purposes of motivation. *(3)* To fix the most essential facts. *(4)* To provide a pacing or timing device for the children to use.

17. In the systematic study of the basic subtraction facts, which of these should come first? *(1)* A problem using the fact. *(2)* Proof for an example such as $7 - 3 = 4$. *(3)* The development of the fact from rearrangement of a group of objects. *(4)* Counting out objects to develop the fact.

18. Should the child's first experience in adding two-place numbers be oral or written? *(1)* Oral. *(2)* Written.

19. What advantage does the form $\begin{array}{r} 2 \text{ tens} \\ + \ 3 \text{ tens} \\ \hline 5 \text{ tens} \end{array}$ have over the form $\begin{array}{r} 20 \\ 30 \\ \hline 50 \end{array}$?

(1) The first harmonizes with the number system. *(2)* The first is similar to the addition of one-digit denominate numbers. *(3)* The first containing fewer numbers is more easily written. *(4)* N.

20. What argument is there for teaching the addition of even tens before teaching the addition of tens and ones? *(1)* No carrying is involved. *(2)* It is less difficult to rationalize the addition of even tens. *(3)* There is no convincing argument. It is merely a matter of opinion. *(4)* N.

21. What is the learning problem to be mastered in the child's first lesson on the addition of two-place numbers? *(1)* How to add first the ones and then the tens. *(2)* How to get the answer. *(3)* How to show or prove that his answer is correct. *(4)* N.

22. Which device, the tens block or the abacus, is most effective in teaching carrying? (*1*) Tens block. (*2*) Abacus. (*3*) N.

23. Is use of the method of making change as performed by clerks good argument in favor of the additive method of subtracting? (*1*) Yes (*2*) No. (*3*) N.

24. Teachers who wish to have their pupils rationalize processes that are being taught find the equal-addition method in subtraction undesirable Why? (*1*) Because adding a ten to the ones numbers is contrary to the notation with which the child is familiar. (*2*) Because adding to the subtrahend is contrary to what the child has learned about borrowing. (*3*) Because the child is not familiar with equations. (*4*) Because adding is the opposite of subtracting.

25. The take-away method of subtraction is favored by most teachers because it is the method they were taught. What other important argument is there in favor of the take-away method? (*1*) It is faster. (*2*) It is more accurate. (*3*) It involves fewer steps when borrowing is encountered. (*4*) N.

26. The writing of the borrowed 1 in subtraction problems is opposed by many teachers. Why? (*1*) It isn't essential to the process. (*2*) It interferes with development of mastery of subtraction. (*3*) It creates difficulties in checking or verifying work. (*4*) N.

27. The exercise in which children indicate the number of people that might live in three or four houses is considered a better exercise for introducing column addition than is the exercise in which children guess what three numbers equal another number. Why? (*1*) Because use of people makes visualization easier. (*2*) Because some numbers are visible in the house exercise. (*3*) Because guessing is not compatible with teaching. (*4*) Because there are too many possible combinations in the guessing exercise.

28. Higher decade addition facts are needed in column addition. Where else are they needed? (*1*) In addition of

three- and four-place numbers. (*2*) In short division. (*3*) In the addition check of subtraction examples of two or more figures. (*4*) N.

29. Which of these higher decade addition facts should receive most attention? (*1*) 26 + 6. (*2*) 29 + 3. (*3*) 23 + 9. (*4*) 24 + 7.

30. What makes such exercises as "Count by 7's to a hundred, beginning at 15" especially good for practice in column addition? (*1*) Long columns of 7, because there are 7 days in the week, are frequently encountered. (*2*) Because every addition involves bridging. (*3*) Because every addition involves a higher decade addition fact. (*4*) Because addition of 7 with every other numeral is involved.

31. Subtraction of three-place numbers is more difficult than subtraction of two-place numbers because three instead of two subtractions have to be made. Is it more difficult in any other way? (*1*) Yes. (*2*) No.

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8 Wilson, G. M., M. B. Stone, and C. O. Dalrymple. *Teaching the New Arithmetic.* New York: McGraw-Hill Book Company, 1939. Pp 101–149.

6

Multiplication and Division

Variations in Present Methods of Teaching the Basic Facts

Current practice is sharply divided on a number of issues relative to the teaching of multiplication and division. In the main, the issues relate to the procedures used in initial instruction, the question of teaching the two processes simultaneously, and the order and number of facts to be included in the first teaching block. Varying practices on initial instruction are well illustrated by the points emphasized in the first lessons in multiplication. Some texts emphasize counting by 2's, 3's, 4's, and 5's; others emphasize the relation between multiplication and adding; while still others emphasize multiplication as something new — number facts similar to and just as important as the addition and subtraction facts. Some texts develop all the facts — for example, all the facts for 2 — before introducing the 3's, others introduce the 2's, 3's, 4's, and so on, simultaneously. Although most instructional programs begin multiplication with the study of the 2's, the 1's and the 5's are used first in some programs. The use of 1 as a multiplier is omitted from some initial instructional programs and emphasized in others. The doubles (2×1, 2×2, 2×3, and so on) are made the foundation for the teach-

ing of multiplication by some texts and are neglected or less emphasized by other texts

Equally divided practice is found with regard to the simultaneous teaching of the processes of multiplication and division. Most textbooks teach first the easy multiplication facts and then, four or six weeks later, introduce the easy division facts. Others separate the teaching of the two processes by only a few days. A few texts introduce both processes at the same time. This is the plan recommended by Morton.[1] Wheat recommends that the division idea be developed first.[2]

Not only is there disagreement as to whether multiplication and division should be introduced simultaneously, but there are also marked differences in the order in which the facts of the two processes are taught and in the rate at which the facts are introduced. In the paragraph above, where variations in initial practice were listed, it was pointed out that some programs begin with the 2's, others begin with the 5's, and still others use the 1's first. In addition to such differences in the order of presentation, the following occur (1) Some programs teach facts without reference to the converse of that fact. For example, $5 \times 3 = 15$ is taught, but the child is not at the same time shown that $3 \times 5 = 15$. On the other hand, some texts emphasize both facts as soon as one is introduced. (2) Most texts present only the so-called easy facts (multipliers and multiplicands of 1, 2, 3, 4, and 5, and quotients and divisors of 1, 2, 3, 4, and 5) in initial instruction. Some texts present all facts through 9 as soon as the group — for example, the 2's — is introduced. (3) In some programs the multiplication and division of 10's is begun as soon as two or three groups (2's, 3's, 5's) have been taught. In other programs all the basic facts are taught before work with 10's is begun.

[1] R L. Morton, *Teaching Arithmetic in the Elementary School* (New York: Silver-Burdett Company, 1937–38), I, 213

[2] H. G. Wheat, *The Psychology and Teaching of Arithmetic* (Boston. D. C. Heath and Company, 1937), p. 300.

Learning Prior to Systematic Instruction

In the number-concept-building exercises described in Chapter 3, the ideas of multiplication and division frequently occur. First-grade children use the terms *two times*, *three times*, and *twice as much*. They count by 10's, 5's, 2's, and 3's. In arranging objects they frequently arrive at a total number, such as 15, by grouping in 5's and then counting 5, 10, 15. During study or use of such exercises, expressions such as "In 10 there are two 5's" and "Three 5's are 15" are heard. The exercise in which pupils indicate the number of people that might live in three houses when the total number is given is another experience through which ideas of multiplication and division might be developed. Many other such exercises may also contribute to a better preparation for this phase of arithmetic.

Since exercises of this type contribute to phases of arithmetic other than multiplication and division, they are not postponed until the child is ready to begin the study of these two processes. Instead, they are a part of the concept-building program which begins in the first grade. Additional background material for multiplication and division is provided in the oral exercises and class projects which confront third-grade pupils. The rather impressive list of learning experiences presented in the preceding section is not incidental to any first- and second-grade program. Such experiences result only from careful planning. Programs which include the features described above are illustrations of the principles of allowing a long time for children to learn and of providing the essential experiences for understanding a process.

Beginning Instruction in Multiplication and Division Facts

Even though the child has had extensive experience in a concept-building program and in solving ordinary quantitative situations of life with the multiplication and division facts, it

is not likely that he will have developed a systematic plan for using them or that he will have gained an adequate understanding of the two processes. There is, then, a real need in the early school experience (preferably the third grade) to teach multiplication and division.

Teachers who follow the plan of teaching recommended by the author of this book begin the study of multiplication and division by assigning several problems involving situations which require use of these two processes for most efficient solution of the problems. The main points of procedure in such an introduction can be presented best by a description of the learning experiences of a class beginning the study of multiplication and division.

"Write the answer to the six problems in this exercise. If you do not know the answer, make a drawing to show what the problem tells with words. Then get the answer in the best way you can. Count if you need to."

These are the directions that were given to the class. Two of the problems used are reproduced here

1. Henry made three trips to the stage, carrying four chairs each trip How many chairs in all did he carry to the stage?
2. We are storing the ten best candles we made in boxes. If each box holds only two candles, how many boxes do we need?

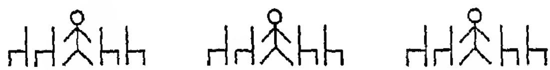
As the children worked on the assignment, the teacher went about the room to locate those who needed help. She made suggestions such as, "Take chairs to the corner of the room. Be sure to carry the same number Henry did and make only as many trips as he did", and she raised such questions as, "Does your drawing show the same thing that the problem told?" After work had been going on for a time, she held up some of the children's papers so that the class could see the ways that were being used to answer the questions. As she showed the papers, she commented on them in order to motivate the work and to give hints to those who seemed to be having difficulty in getting

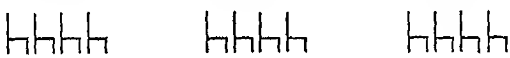
the idea. For those who were able to answer the questions without the use of diagrams and therefore finished in a relatively short time, the following assignment was given:

1. With marks or a drawing show that your answer to problem 1 is correct.
2. Show with numbers what you showed with a drawing.

The same assignment was given for the remaining five problems. Those who finished their work quickly were also encouraged to find other ways of showing that their work was correct.

During the class period the teacher had children place representative diagram and number solutions on the board. An attempt was made to get different ways of showing how the answer was obtained. The following are representative of the work of the children:

(a)  12

(b)  12

(c) | | | | | | | | | | | | 12

Trip 1 Trip 2 Trip 3

(d) | | | | | | | | | | | |

4 8 12

(e) The actual chair arrangement was also used — that is, the three groups of four chairs each were placed at the front of the room.

Most of the number solutions were simply the addition of three 4's. One child offered

$$\begin{array}{r} 4 \\ \underline{3} \\ 12 \end{array}$$

but this was rejected by the pupils because they said it looked too much like adding 4 and 3.

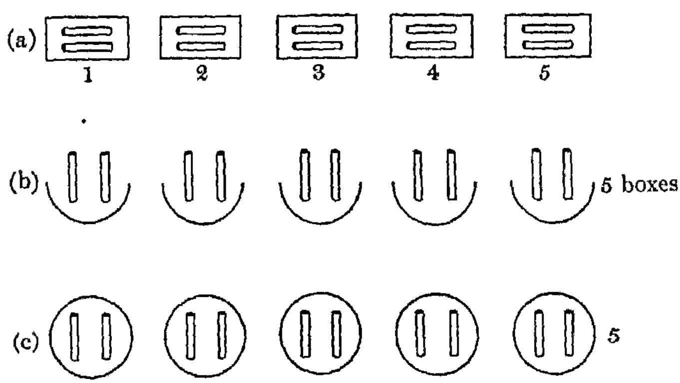
In the discussion of solutions these questions were asked. "Does the solution show what the problem told? Does it answer the question?" In some cases, notably solutions *c* and *d*, there was a question about the validity of using marks for chairs. The teacher answered by saying that marks were quite all right if the person using them knew that they represented the chairs.

At the close of the discussion of the "number solutions" the teacher said, "Here we have shown the same number fact in several ways. Who can tell me what the number fact is that has been demonstrated?"

"Three 4's are 12" was selected as the best statement of the fact. This statement and the statements of other facts were written on the board. The teacher remarked that these number facts would be used again and again.

The diagram and number solutions for problem 2 appear below. For convenience, problem 2 is repeated here.

"We are storing the ten best candles we made in boxes. If each box holds only two candles, how many boxes do we need?"



Not all the children offered number solutions, but the following are representative of those offered:

$$
(a)\;\begin{array}{r} 10 \\ \underline{2} \\ 5 \end{array}
\qquad
(b)\;\begin{array}{r} 10 \\ \underline{-\;2} \\ 8 \\ \underline{-\;2} \\ 6 \\ \underline{-\;2} \\ 4 \\ \underline{-\;2} \\ 2 \\ \underline{-\;2} \end{array}
\qquad
(c)\;\begin{array}{rl} 2 & (1) \\ 2 & (2) \\ 2 & (3) \\ 2 & (4) \\ \underline{2} & (5) \\ 10 & \end{array}
$$

Number solution *a* was explained by the statement: "Because in 10 there are five 2's." The solution was rejected, however, because "it looked too much like addition." Solutions *b* and *c* were accepted, although *c* was questioned by the teacher because it showed that five 2's make 10, not how many 2's in 10, which was the original situation raised by the problem. The teacher called attention to the length of the two solutions accepted.

The children experienced difficulty in trying to make a short statement for the number fact of their solutions. "In 10 there are five 2's" and "five 2's are 10" were statements they most frequently offered. The similarity between the second statement and the statements for the fact of problem 1 was noted. When the children decided that the two were different types of problems, the first statement was selected as the better one, and the teacher wrote the second statement in parenthesis just beneath the first. The importance of writing this second statement will be seen when the question arises of how to justify the long-division form as a record of thought.

In the next several assignments for this class, problems involving other multiplication and division facts were used, and facts were developed in the manner described above. For a short presentation of the actual procedures, see sample program III, Chap-

ter 2. In that sample only multiplication was used. The steps for division, however, would be very similar. In the introductory procedures just described, several points are important enough to warrant separate consideration. Instead, then, of considering further the learning experiences of one class, the remainder of this section is given to a discussion of special issues in the teaching of multiplication and division.

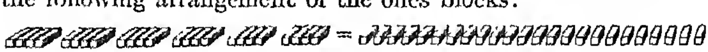
In the teaching situation outlined above, the processes of multiplication and division were introduced simultaneously. While the two are related processes and consideration of them together may make for better understanding than as if they were treated separately, there is no evidence to support this conclusion. Simultaneous treatment has the advantage of providing a longer period of time between introduction and learning for mastery than would be true if separate teaching were done. On the other hand, many teachers see an advantage in concentrating on one process. They claim that the value of clear and forceful introductions to processes is lost if many other facts and even other processes are brought in before the child has an opportunity to master the first fact introduced. The question of simultaneous or separate treatment of multiplication and division is, then, a matter on which it is impossible to make a recommendation that is backed up by evidence from research or unanimity of practice. Since children work at different rates, it is obvious that not all the children in the class referred to above finished the assignment at the same time. It should be noted that some procedures used in the program were included primarily to give the slower pupils more time — for example, using several problems and suggesting that pupils try to discover more than one way of finding the answers. Additional work was introduced which consisted: (1) of showing with tens blocks and ones blocks that such statements of fact as "four 3's make 12" are true; (2) of making additional problems to illustrate the facts, (3) of finding other ways of proving; (4) and of making a

list of all the forms of proof for one fact. Even with extensions of time, not all the problems were solved and illustrated by the slowest workers. Every pupil, however, had an opportunity to try to solve at least one multiplication and one division example before an explanation was given Even if some pupils do not solve any problems, the author believes that they are in a better position to profit from an explanation than if they were given the explanation before trying to solve the problem.

The use of tens and ones blocks shows especially well that multiplication is a regrouping process. In the class referred to above, every fact was demonstrated at least once with these materials. The value of the blocks in showing how multiplication makes for simplicity in expressing quantities is shown in the illustration for the problem, "How many sandwiches do we have if each of the six boxes contains four sandwiches?" The sandwiches in each box were represented by four single blocks. The complete semi-concrete representation then appeared thus:

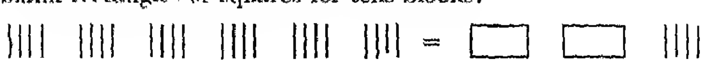
|||| |||| |||| |||| |||| ||||.

To show the total by putting all the ones blocks in a single group answers the question "How many?" but not in a way that makes it easy to recognize the total and to name it in the standard terminology of the number system. This fact is shown by the following arrangement of the ones blocks:



On the other hand, the total can be recognized easily and can be named in standard terms when it is shown by tens blocks and ones blocks:

From such an arrangement of blocks, it is an easy step to representation on a blackboard by marks for single blocks and blank rectangles or squares for tens blocks:



The simplicity of this representation as compared to the first is obvious.

There may be some who will want to argue that the sandwiches could never be grouped that way, and that, therefore, the representation is misleading. To hold thus closely to the concrete would eliminate practically all the uses of number. The abstractness demonstrated in the illustration represents the way in which number renders its service to man. By dealing with symbols limited in meaning, a person is left free to do the necessary reflection. Besides, the question in the problem asks for the total, which means that the reader's attention is directed not to each individual sandwich but to the aggregate. Such rearrangement of groups into tens and ones as that used in the illustration will show children the meaning of multiplication and also its function — that it is easier to think of twenty-four than it is to think of six fours.

Explanation of the nature and function of division can also be made easier through the use of blocks or other objects. For example, consider the problem, "If twenty-one pieces of candy are to be divided equally among seven girls, how many pieces will each girl receive?" In accordance with the basic principles of our number system, the simplest representation of the quantity 21 with blocks would be 

an arrangement of three groups not all of the same size; but to show the share of each girl requires a rearrangement into seven groups of the same size. In other words, if each girl is to receive one-seventh of the total, the entire amount must be composed so that it may be separated into seven parts, each of which will contain the same number of units or of units and parts of units. To divide the quantity into seven parts, the pupil will have to change the two tens blocks into an equivalent number of ones. Then by distributing the single blocks into seven equal groups, he performs the division. This regrouping makes it easy to see

each girl's share and makes clear to the pupil the meaning of this type of division. In order that children may not miss the simplification role played by division, the teacher should call attention to it.

Thus, through the study of illustrative problems, the children not only are able to identify the multiplication and division facts, but they are able also to learn the functions of each process and to see that each is a regrouping process.

After the introductory lessons additional multiplication and division should be developed through the solution of problems. Not so much time should be required for discussion, and, as the children become more familiar with the procedure, not so much time should be required for actual solution of the problems. Teachers should be on guard, however, against the adult's tendency to generalize; that is, to assume that because two or three of the multiplication facts for the fours have been demonstrated, the others may be written on the board by the teacher. The most difficult task of the teacher during this period of instruction is to keep from giving too much aid. To challenge pupils and to direct thinking, the teacher may often ask children who have a wrong number solution or an erroneous diagram to put their work on the board or show it to some other pupil to see if that pupil will accept the solution. A procedure similar to that described in Chapter 5, page 126, will be of value in centering pupils' attention on the limitations of proposed records of thought. The pupils with the help or direction of the teacher should do the evaluating, not the teacher alone. It is only as a last resort that the teacher points out errors. "Do not be in a hurry and thereby rob the child of the thrill of discovery" is a warning that all instructors should heed.

As each new fact is demonstrated, it is written on the board. Following the pattern set in the first discussion, the multiplication facts are written in one column and the division facts in another. The facts should appear in the order in which prob-

lems are discussed; because the problems do not present facts in any special order, the facts will not be arranged systematically. After a number of facts have been demonstrated, the issue of a shorter way of writing the facts is raised.

The need for a shorter or better way of writing the multiplication facts might be introduced in the following manner: "When you first used addition facts, you wrote them this way, '4 and 3 equal 7.' Do you still use that long form?"

Of course, the $4 + 3 = 7$ or $\begin{array}{r} 4 \\ +3 \\ \hline 7 \end{array}$ are considered more convenient ways of writing than the long form. The task may become, then, a case of trying to write the multiplication facts without words. The need for a better form may also be raised by calling attention to the mistake that might be made if "3 4's are 12" were read "34's are 12." If none of the children know the accepted arithmetical way of writing multiplication facts, a textbook is a logical source to consult. Even though the forms $3 \times 4 = 12$ and $\begin{array}{r} 4 \\ \times 3 \\ \hline 12 \end{array}$ are adopted and the children express the fact in words, as "three times four are (or equal) twelve," the expression "Three fours are twelve" should be continued during the period of initial instruction. This language more nearly represents what is actually done. The form $\begin{array}{r} 4 \\ \times 3 \\ \hline \end{array}$ is introduced almost as soon as is "3×4" in most children's texts. An important point for teachers to watch in use of the form $\begin{array}{r} 4 \\ \times 3 \\ \hline \end{array}$ is to see that children read this, not just as three times four but as a question, "How many are 3 4's?" or "3 4's are how many?"

In finding a way of writing a division fact without the use of words, textbooks will be of little value, for few texts give the two statements that the children will have written. It should

be recalled that the two statements are: "In 10 there are 5 2's" and "5 2's are 10." Since the formal horizontal form $10 \div 2 = 5$ omits one of the statements and because it uses the little-understood symbol which is read "divided by," it is for the time being rejected. Instead, the form $2\overline{)10}$ is used for the question, "How many 2's in 10?" and the answer 5 is written above the line to complete the first statement, "There are 5 2's in 10." Then, in order to include the second statement, a 10 is written underneath the original 10. The complete question and statement

$$\begin{array}{r} 5 \\ 2\overline{)10} \\ \underline{10} \end{array}$$

is read, "How many 2's in 10? Five. Five 2's make 10." Through this procedure the child is given a reason for using the long-division form and a useful relationship is emphasized.

It is recommended that a fair-sized group (twenty to thirty) of the multiplication facts of the 2's, 3's, 4's, and 5's be introduced before study for automatic mastery is attempted. To prepare for this later stage of the instructional procedure, a period should be taken to arrange the facts in systematic order. This will result in the familiar multiplication table, or at least a part of it. The division facts are not so easily organized and the organization is not so useful. Nevertheless, a table form of organization seems worth using. When the facts to be mastered have been developed, a procedure similar to that used in learning the basic addition and subtraction facts is followed. Briefly, this procedure involves the creation of a situation which demonstrates the value of knowing the facts, a plan of concentrated study, and the study period itself interspersed with procedures which test the children's understanding of the facts being memorized. For a complete description of the learning procedure, see "Method of Learning," Chapter 2, and the topic, "Intensive Study of the Basic Addition and Subtraction Facts," Chapter 5.

Order of Teaching the Basic Multiplication and Division Facts

The basic facts involving 2, 3, 4, and 5 as multiplicands and divisors are usually referred to as the easy multiplication and division facts. The facts involving 6, 7, 8, and 9 are called the harder multiplication and division facts. The statement has already been made that the children themselves should develop tables of these facts. For study exercises and for test purposes, the facts in table form have little value. For such purposes an arrangement that presents the combinations or number questions in mixed order is far more useful. An example is presented below

EASY MULTIPLICATION COMBINATIONS

3	2	3	5	2	2	3	3	3
2	5	6	6	2	3	4	3	9
2	3	4	4	5	2	3	2	2
9	8	5	7	8	6	7	4	8
2	3	4	5	5	4	5	5	4
7	5	4	3	2	6	7	4	3
5	4	4	5	4				
5	8	9	9	2				

EASY DIVISION NUMBER COMBINATIONS

$2\overline{)4}$	$4\overline{)8}$	$2\overline{)8}$	$3\overline{)9}$	$4\overline{)12}$	$4\overline{)20}$	$3\overline{)6}$	$5\overline{)30}$
$5\overline{)10}$	$2\overline{)6}$	$3\overline{)12}$	$4\overline{)16}$	$3\overline{)15}$	$5\overline{)15}$	$5\overline{)35}$	$2\overline{)10}$
$2\overline{)12}$	$5\overline{)45}$	$4\overline{)32}$	$3\overline{)18}$	$3\overline{)27}$	$2\overline{)18}$	$4\overline{)36}$	$2\overline{)16}$
$5\overline{)40}$	$4\overline{)28}$	$5\overline{)25}$	$3\overline{)21}$	$2\overline{)14}$	$3\overline{)24}$	$4\overline{)24}$	$5\overline{)20}$

Many lists include the facts involving 1 in the list of easy facts. They are not included in the lists above because children have little use for them until multiplication of 10's is begun or until division problems involving quotients greater than 9 are encountered. Furthermore, it is difficult to teach multiplication by 1 in a meaningful way. Considerable difficulty is also encountered in trying to present with meaning the division facts having a quotient of 1.

The harder multiplication and division facts include the 6's, 7's, 8's, and 9's, and of course the 1's if they are not included with the easy facts. A list of the harder combinations is given below:

HARDER MULTIPLICATION COMBINATIONS

7	7	7	6	6	9	9	7	6	7	
2	3	1	2	4	2	8	7	5	5	
8	9	9	6	6	8	9	8	8	6	8
1	4	6	6	1	3	7	8	5	9	2
9	9	6	7	6	7	9	9	8	6	8
9	1	3	6	7	4	5	3	4	8	9
7	7	8	8							
8	9	7	6							

HARDER DIVISION NUMBER COMBINATIONS

6)12	7)49	8)40	6)6	7)21	9)18	8)8	7)28	6)30
6)42	7)49	8)40	9)72	6)48	7)42	8)16	7)7	9)36
9)54	8)32	6)24	9)81	9)63	8)56	9)45	9)9	7)14
6)18	6)54	7)63	7)35	8)48	8)72	9)27	8)24	8)64

Notice that no zero facts are included in this list. (A method of handling zero in both multiplication and division will be discussed in the treatment of multiplication of tens and in division where quotients include zero.)

Just as was true in the case of the addition and subtraction facts, the order of teaching the multiplication and division facts is not a crucial issue. All the facts are so important that they must be mastered. According to present practice, the major part of this learning is assigned to the third and fourth grades. It has been demonstrated again and again that children of this mental age can understand and memorize the basic facts of multiplication and division. Little will be gained by postponing the introduction of the difficult facts to the last weeks of the fourth grade. The increased mental age of the children will hardly compensate for the shortened period of study.

The Long Form Versus the Short Form of Division

In the preceding description of the teaching of the basic division facts, the long-division form rather than the more commonly used short-division form was used. The form $\begin{array}{r} 6 \\ 2\overline{)12} \\ \underline{12} \end{array}$ is the long form of division, while $\begin{array}{r} 6 \\ 2\overline{)12} \end{array}$ or $\begin{array}{r} \underline{2)12} \\ 6 \end{array}$ is the short form. The difference between the two forms becomes more obvious if a larger dividend is used. The following examples illustrate this point.

$$\text{Short Form: } \begin{array}{r} 147 \\ 3\overline{)441} \end{array} \qquad \text{Long Form: } \begin{array}{r} 147 \\ 3\overline{)441} \\ \underline{3} \\ 14 \\ \underline{12} \\ 21 \\ 21 \end{array}$$

All well-known authorities now advocate teaching the long form of division before the short form. Even though practically all textbooks since 1935 state that the long form of division is to be taught first, they present the basic facts in the short-division form. The long form is usually not introduced until quotients with remainders or two-place quotients are introduced.

In this book the decision in favor of using the long rather than the short form for beginning work in the writing of division examples is based upon logical analysis of the difficulty of the two processes and on research data relative to use and difficulty. The writer not only recommends the teaching of the long form of division before the short form, but also recommends that short division be taught only as a short cut. As was indicated in the teaching procedure, the long-division form is recommended for even the basic facts. This procedure is suggested for its intrinsic merit and because teachers have found it difficult to get children who have used the short form with the basic facts to use the long form when easy, two-place dividends are used as introductory material for teaching the long-division process. Children who have been using the short form often fail to see why the teacher insists that the long form should now be used. If, on the other hand, the long form is used from the beginning, this particular point of confusion is avoided.

Multiplication Tables

A point in the teaching of the multiplication facts that needs clarification is the use of tables. Arithmetic instruction has been severely criticized because of the manner in which tables have been used in the teaching of the multiplication facts. This criticism has been just when directed at the meaningless memorization of the facts in table form. Too often, before the meaning of the facts has been developed, the assignment has been to "learn the 4's for tomorrow," or the classroom procedure has

consisted of having pupils take turns reciting the tables while the other members of the class listen for errors. For such use of tables, there should be only words of condemnation.

There is, however, a very useful role that tables can play in learning the facts. After multiplication facts have been illustrated through problems and the nature of facts demonstrated through use of objects and marks, a systematic arrangement of these facts in tables is desirable. Any organization (system) that makes for easier learning — whether that learning be done through the more ready visualization of relationships, through the use of facts already learned, or through some other advantage made possible by organization — should become a part of the instructional program. The method of instruction advocated in this book puts emphasis on the use of relationships. The multiplication table expresses in an efficient manner the relations among the multiplication facts. For pupils who understand the meaning of these facts, there is no harm in studying the facts in the serial order of appearance as in the table.

The serial order of learning tables is criticized because pupils supposedly go back to 1×7 and say all the intervening "7" facts in order to give the answer to 6×7 correctly. A few children probably do on occasion go through such a long and tedious procedure to get one fact. In fairness, however, it should be pointed out that even the critics of the procedure admit that the child gets the correct answer. On the other hand, many children who study by the isolated-fact method are never able to give the answer to 6×7. A long way of getting an answer is better than none at all. Practically all adults, in working long problems, sometimes reach the end of their attention span or momentarily forget what 6×7 equals. Few of them go back to 1×7, but they do refer to 5×7 or 7×7. For such related facts, the table form of organization is ideal. Multiplication tables are recommended, therefore, as an integral part of the learning materials of arithmetic. Every child should

make his own table and use it in learning or fixing the facts. As a check on understanding, children can be required to show the truth of facts they learn from a table.

Measurement and Partition in Division

Two division problems used in the discussion of beginning instruction, pages 173 and 179, illustrate the two kinds of division commonly referred to as measurement and partition. The first — "We are storing the ten best candles in boxes. If each box holds only two candles, how many boxes do we need?" — is an example of measurement division. Here the operation is very definitely a case of finding how many two-candle measures can be filled from a total stack of ten candles. In other words, one dips into the supply, filling the desired measures until the supply is used up. From this analysis, it is easy to see why the name *measurement* has been applied to this type of division problem. The problem, "If twenty-one pieces of candy are divided equally among seven girls, what is each girl's share?" is an example of partition division. To find the answer in this problem each girl's share or part is determined.

The thought process required in the solution of problems in measurement division differs from that required in partition division. Procedures correspondingly different are also followed by children in explaining with objects how to solve problems representative of the two types of division.

In order to show just what difference in thinking occurs, two problems using the same division fact will be used.

1. How many 3-cent stamps can be bought with 15 cents?
2. If 3 stamps cost 15 cents, how much will each stamp cost?

The thinking that children (and adults also) employ in solving the first problem will take one or the other of these two courses: (*a*) To find how many 3-cent stamps I can get for 15 cents, I find how many 3's are in 15; (*b*) I'll get as many stamps

as the number of 3-cent piles I can make from this pile of 15 cents.

Thinking as in *a*, one adds 3's until 15 is obtained for a total and then counts the 3's used. Children demonstrate the *b* kind of thinking when they put down 15 marks and then circle 3 marks at a time until all are grouped. This type of problem is then clearly a case of measuring to find how many measures of a given size (3 cents) are contained in the total (15 cents).

The thinking used in solving problem 2 may be of these two types: (*a*) To find the cost of each stamp, I'll put one cent at a time on each stamp until all the 15 cents are used. (*b*) Since there are 3 stamps, each stamp will cost one-third of the total, or one-third of 15.

Study of the illustrations of thinking used in solving the two types of problems will show that the thought-processes are different and that that required for the partition-type problem is more difficult. In measurement problems the task is to find the number of parts or measures contained in the total while in partition problems the task is to find the size of *one* part.

It is obvious that no difference in solution or thinking will occur if the children are taught to solve partition division problems such as this by finding the number of 3's in 15. If such a plan is followed partition problems should not be used to illustrate the division process, for the partition problem does not require finding the answer to such questions as how many 3's in 15. For further discussion of this fundamental difference, see section on "Methods of Learning," in Chapter 2, and section on "Purpose of Problems," in Chapter 7.

Since the thinking required for the solution of partition problems is more difficult than that required for measurement, the latter type should be used in the first illustration of division facts. Since partition problems occur so frequently in life, however, this type should also be used before children have completed the systematic learning of the division facts. It should

be noted that the common form of writing division number questions such as $4\overline{)12}$ is not applicable to partition problems. The foregoing number question asks, "How many 4's in 12?" — a measurement question. In partition, the question is "One-fourth of 12 is how many?" or, "12 divided by 4 is how many?" Thus, when partition problems are introduced, a logical reason for introducing the form $12 \div 4$ is provided. Following introductory work in division the distinction between the two forms of division is of no significance. The two forms are interchanged in much the same manner that *multiplier* and *multiplicand* are interchanged in multiplication

While it is not necessary to acquaint children with the difference between measurement and partition, teachers should certainly recognize the difference if they are to be of service in guiding the thinking of children. The need for understanding the distinction will be especially evident if children are asked to prove their work. Then, too, both types of thinking are prerequisite to an understanding of the division of fractions. A specific example of a difference in teaching that is brought about through knowledge of measurement and partition is found in the initial teaching of the division of tens (see page 199).

Summary of Procedures Used in Learning Basic Facts

After a long period of concept-building and after thorough study of counting, addition, and subtraction, the multiplication and division facts are introduced through the use of problems. The facts discovered or demonstrated by the children from diagrams and number solutions are identified and written on the board. In class discussion the best ways of writing and organizing the facts are selected. When all or a large part of the basic facts with 2, 3, 4, and 5 for both multiplication and division have been demonstrated and identified, a test is given to show in what way automatic mastery of facts is of value. A discus-

sion is then devoted to methods of learning the facts for automatic mastery. The most important of these learning procedures are: (1) studying facts from flash cards; (2) studying facts given in textbook, in workbook, on the board, or on individually prepared practice sheets, (3) writing the facts; (4) studying the multiplication table and division table; and (5) taking tests on the facts.

During this period of intensive study, children are frequently asked to show their understanding of the facts and their understanding of the processes by making drawings or other forms of illustration.

Learning 6's, 7's, 8's, and 9's

If the learning of the 3's, 4's, and 5's is completed in the third grade, the learning of the remaining facts is the first new major task of the fourth-grade child. After a review of the previously learned facts, these harder facts are studied by a method quite similar to that used in learning the easier facts. Because all products except the first one with each number are a ten and one number, perhaps a little more emphasis is given to the idea that multiplication is a process of rearranging equal groups of one into 10's and 1's than was the case with the 2's, 3's, 4's, and 5's. However, not as much time is needed in the demonstration phases since the pupils are already familiar with the multiplication procedure and with ways of showing facts. The same study procedures recommended for learning the easy facts will prove useful for studying the 6's, 7's, 8's, and 9's.

Remainders in Division

The problems used in illustrating the basic facts of division come out even. In life, however, there are far more problems with remainders than without. The exclusive use of problems without remainders during the period in which the basic facts

are being learned is justified on the ground that the fewest possible difficulties should confront the learner of a new process. Soon after the period of intensive study of the first group of facts, however, problems which involve remainders should be assigned. If these are problems which the children understand, they will offer various logical ways of showing the amount left over. For example, in a problem involving the division of seventeen sandwiches among four boys, one child suggested that the "extra one" be given to one boy's mother, because she had made the sandwiches. The fact that each boy can share equally in this left-over sandwich can be brought out by using pieces of paper or other divisible materials to represent the sandwiches.

The problem of how to write this divided unit is one that should receive the attention of every member of the class. Since a fraction is an indication of division, the best method is probably the fractional form. For problems such as the division of sandwiches, where the pupil can readily see that it is sensible and practicable to divide the remainder into smaller parts, children should begin to use this form in the third grade. There are, however, some situations, such as the division of marbles, where the division of the left-over part or remainder would be foolish. Children must, therefore, also learn to write remainders as R = 2, or 2 left over.

The solution of problems with remainders also will demonstrate the need for trial quotients. Since the dividend of such a problem will not be a number that occurs in a basic fact, the child will resort, of course, to the best-guess method of determining the quotient. Although this best guess should be an intelligent one, based on a "near" basic fact, there are times when the guess or trial quotient will be wrong. Children should be helped to face this situation intelligently — that is, to consider whether the trial quotient was too large or too small. For example, the child confronted with the task of determining the

number of 3's in fourteen should be led to think of numbers close to fourteen that contain an even number of 3's. Since twelve and fifteen are two such numbers, the child should base his estimate of the number of 3's in fourteen on the number of 3's either in twelve or in fifteen. If the latter is used, the estimate will, of course, be wrong, but it is from such an experience that the child learns to use the smaller of the two near numbers. As soon as children understand the process of division with remainders, an intensive practice period should be undertaken. Here, as in other phases of arithmetic, sets of examples are extremely useful. A sample set is given below:

DIVISION EXAMPLES REQUIRING REMAINDERS

$2\overline{)15}$	$3\overline{)22}$	$4\overline{)9}$	$5\overline{)16}$	$3\overline{)10}$	$2\overline{)11}$	$4\overline{)14}$
$6\overline{)15}$	$3\overline{)11}$	$4\overline{)13}$	$5\overline{)22}$	$3\overline{)17}$	$2\overline{)9}$	$4\overline{)21}$
$7\overline{)15}$	$2\overline{)17}$	$2\overline{)13}$	$4\overline{)25}$	$6\overline{)20}$	$4\overline{)23}$	$5\overline{)19}$
$4\overline{)19}$	$5\overline{)27}$	$6\overline{)16}$	$3\overline{)16}$	$6\overline{)26}$	$7\overline{)18}$	$3\overline{)22}$

Because of the need for interpreting the remainders, problems should be used extensively during this period.

Multiplication of Two-Place Numbers

When children, in learning to count, discover that twenty is 2 tens, thirty is 3 tens, and so on, they are laying the foundation for multiplication of two-place numbers. This idea is further developed when children find on the number chart the number that means 7 tens or 9 tens. In both these procedures, the idea basic to multiplication of two-place numbers is implied if not actually used. This is the idea that tens are handled just as ones are. For example, the fourth ten (forty) comes just after the third ten (thirty), just as the fourth one comes after the

third one, and the fourth ten is composed of four things just as the fourth one is. The teaching of the multiplication of tens is not undertaken, however, until after the children have had an opportunity to learn at least the first group of the basic facts.

The actual instructional procedure is similar to that for teaching the basic facts and therefore does not have to be described again. The first problems should involve the use of even tens only and not tens and ones, for example, 4×20, not 4×21. This recommendation is not in harmony with the procedure given in most children's texts and is not recommended by most writers on the teaching of arithmetic. If the idea of ten as a collection has been developed and of zero as a place-holder, the use of tens only should be easier than the use of tens and ones. First problems, then, should involve the multiplication of 20, 30, or 40, and not 21, 22, 31, 32, and so on.

Children will not immediately employ the multiplication idea as it should apply to tens. For example, in solving 8×20, a favorite procedure is to double 20 and then double that and then add the two 80's. This doubling and adding is done without the use of pencil and paper. Written number solutions are usually of these types:

(*a*)	(*b*)	(*c*)
$\begin{array}{r} 20 \\ \times 8 \\ \hline 160 \end{array}$	$8 \times 20 = 160$	$\begin{array}{r} \text{2 tens, no ones} \\ \times 8 \\ \hline \text{16 tens, no ones} \end{array}$

Here the children evidently follow the pattern they have learned in studying the basic facts. Their oral solutions are far more valuable in the discussion of methods of solving multiplication problems than in previous learning procedures. For example, in telling how they know that $8 \times 20 = 160$, children usually say, "8×2 tens = 16 tens and that's 160." The word-presentation used in solution *c* emphasizes the idea that it is multiplication of tens.

Note that solution *a* is the first case in which a so-called "zero fact" in multiplication is encountered. If the children do not

raise the question concerning whether the 0 is multiplied by the 8, the teacher should. In helping the children answer, the teacher should refer to word solutions or to one of the oral solutions. Since the zero was not multiplied in those solutions, the pupils may conclude that zero is not needed in the pencil and paper number solution. Such a conclusion has some merit and should receive careful consideration. In this consideration, the fact should be brought out again, preferably through a question, that the zero is holding a place and that it is used because there are no ones in the number. Therefore, in order to keep a person from mistaking the 2 for ones, the zero is used in the ones place. Since zero is not a numeral representing quantity, there is no quantity there to be multiplied.

If zero is not multiplied, what is the actual procedure that the children are to follow? Ideally, they are to think, "There are no ones: therefore, I put a zero to hold the ones place." The situation in most classrooms is, however, far from ideal. Therefore, most teachers will find that it is advantageous to show that zero can be multiplied. In fact most adults use $8 \times 0 = 0$ in exactly the same manner as $8 \times 2 = 16$ is used. While this use of a verbal statement such as "eight times zero equals zero" may be of value in teaching the multiplication procedure, it can hardly be claimed that such statements are essential to understanding.

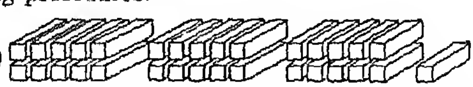
The introduction of the multiplication of zero as presented in the preceding section is then a part of the introduction to the multiplication of tens. In such situations a need for multiplication of zero is created. After several zero facts have been presented, development of the generalization that zero multiplied by any number equals zero should be fostered. Just as is true of other arithmetical procedures, introduction through demonstration of a need, explanation of the facts, and development of a generalization will not once and for all solve the multiplication of zero by an integer. For most children there has to

be systematic practice. The use of cards on which zero combinations and facts appear has a place, although not nearly so important a place as such cards have in the learning of other multiplication facts.

The multiplication of an integer by zero and zero by zero should be taught in a manner similar to that outlined above for the multiplication of zero by an integer. The multiplication of an integer by zero, of course, will not arise until multiplication with tens is undertaken.

From the various number solutions, the children should eventually choose the type $\begin{array}{r} 20 \\ \times\ 8 \\ \hline 160 \end{array}$ as the best form. As a result of the evaluation and discussion of solutions, two conclusions should be reached. First, the best method of solution should be selected and labeled "to be learned"; and second, the generalization that tens are multiplied just as ones are, should be formulated and accepted. In connection with this generalization the following precaution should be stressed: A zero must be used to hold the ones place, lest the tens be read as ones.

Following work with tens only, problems involving the multiplication of tens and ones (no carrying) should be presented to the class. As in the previous introduction of new phases of arithmetic, the method of confronting the child with a problem situation should be followed. Since the solution of a problem involving tens and ones illustrates a new procedure, some of the work of children in such a situation is shown below. To solve a problem involving 8×31, one group of children suggested the following procedures:

(a) 

$$\frac{\text{(tens blocks)}}{\text{(24 tens)}} \quad \frac{8}{8} = 248$$

(*b*) 8 31's added

(*c*)
$$\begin{array}{r} 31 \\ \times 8 \\ \hline 248 \end{array}$$

(*d*)
$$\begin{array}{ll} 3 \text{ tens} & 1 \text{ one} \\ \times & 8 \\ \hline 24 \text{ tens} & 8 \text{ ones} \end{array}$$

After the procedures for multiplying tens and tens and ones have been discussed and evaluated and the class seems to have a good understanding of them, class time should be used in considering the question, "How does the multiplication of tens differ from the multiplication of ones?"

As soon as a satisfactory answer has been agreed upon, the teacher should call the attention of the children to the fact that new procedures are quickly forgotten unless precautions are taken to fix them in mind. Intensive work on examples should then be undertaken. During this intensive practice period, some problems may be used to furnish a setting for the proofs that are constantly needed as a check of understanding. This requiring of proof for problems is more than a test procedure. Through such exercises the child finds and fixes new relationships. The requirement of proof also affords an opportunity to show the child the limitations of his knowledge. If, for example, the child has only a superficial idea of what he is doing, the questions that the teacher or other pupils ask about his proof will quickly show him his limitations. Addition, use of tens blocks or diagrams, and various forms of doubling are examples of acceptable types of proof. Further work with multiplication of two-place numbers will be found as a part of the maintenance and review exercises in all texts.

Following the work described in the preceding section, multiplication of two-figure numbers involving carrying should be introduced. Here, as in previous introductions to new processes, a problem situation should be used in initial instruction.

Since the children will already be familiar with carrying in addition, this new use of the process will not be difficult. Children usually find that the tens blocks and ones blocks or the bundles of sticks are the best means of showing what they do when they carry in multiplication. A solution of the type used in example *d*, page 197, has also proved a valuable aid in showing why carrying is essential.

After the process of carrying is understood, a period of intensive study is essential to fix the process in the minds of the children. For this intensive study, examples are the most economical means of practice. However, to maintain interest and to provide the essential setting for situations where proof is required, some problems may be used. A set of examples of the types suitable for practice is given in the list:

18	26	17	24	22	35	23	35
2	3	3	4	5	5	4	4
23	35	42	33	25	12	22	34
6	6	7	6	8	9	8	6

The examples above may be used as guides in constructing problems.

Division of Tens

In learning the basic division facts some division of tens numbers occurs. However, in such division the tens numbers are always divided into ones (e.g. $32 \div 4 = 8$). This section is concerned with the division of tens numbers into tens or tens and ones. The division of such tens will introduce the use of two-place quotients (e.g. $70 \div 2 = 35$).

Like other new work, the procedure is introduced through use of a problem situation. The first problem used should involve only quotients of even tens, such as 10, 20, 30. In order to demonstrate the fact that tens can be divided just as ones are, only partition problems should be used in the introductory

lessons. For example, solving the problem, "If 3 pencil sets cost 60 cents, how much does each set cost?" by the use of tens blocks will show that each part will be two tens or twenty. When this solution is transferred to or written with numbers, the fact that we deal only with the tens is more evident. On the other hand, when the problem, "If one stamp costs 3 cents, how many stamps can you buy for 60 cents?" is solved by means of objects, the 60 must be presented as ones; otherwise, the measure 3 (cents) cannot be applied. (See also "Measurement and Partition.") Children's thinking in solving the partition problem would run somewhat as follows: "Each of the pencil sets would cost one-third of six tens, or six tens divided by three. The answer is two tens." They then should show how to write the process with figures:

$$\begin{array}{r} 20 \\ 3\overline{)60} \\ \underline{60} \end{array}$$

A number of problems should be solved with tens blocks and with numbers before problems involving quotients with tens and ones are introduced.

The introductory problems in which quotients tens and ones occur should also be of the partition type and should be solved by such indirect means as with the tens and ones blocks before a number solution is undertaken. If a measurement problem were used, the diagram or other indirect solution would almost certainly show the changing of tens to ones. Such a procedure is of little value in working from the indirect to the accepted number solution The operation with numbers presents no new features other than the second division. For example, in 36 ÷ 3, the first step of finding how many 3's in three tens has already been learned, as have the multiplication and subtraction processes and the comparison of the divisor with the remainder. In previous division examples, when the remainder was found to be larger than the divisor, a new quotient figure was tried.

In this case, however, a second divisor is needed. To make this second division in the same example clear to children, careful work with the tens blocks or their equivalent should be done by every child. Just one demonstration to the class is not sufficient. It should be remembered that in much of this early work pencil and paper are used primarily to make a record of solutions. This function of written work, plus the fact that children have learned that number solutions aid in simplifying work, are reasons that teachers can give in explaining why the pencil-and-paper method is important enough to be learned.

Teaching the division of hundreds follows very closely the plan just described for tens. The division of hundreds introduces one new difficulty — that of a zero in the tens place of the quotient. Here, as in previous lessons, the exact reason or need for the zero should be discovered or demonstrated by means of a problem in which collections of objects or marks may be used in showing how the actual division can be performed. The solutions of the following problem will illustrate one way of showing this need:

"There are 315 handbills to be distributed by three boys. How many handbills must each boy take if they are each to distribute the same amount?"

(*a*)	First Boy	Second Boy	Third Boy
	Hundred	Hundred	Hundred
	$\mid\mid\mid\mid\mid$	$\mid\mid\mid\mid\mid$	$\mid\mid\mid\mid\mid$

$$(b)\qquad \begin{array}{r} 100 \\ 3\overline{)300} \\ \underline{300} \end{array} \qquad\qquad \begin{array}{r} 5 \\ 3\overline{)15} \\ \underline{15} \end{array}$$

$$(c)\qquad \begin{array}{r} 100 \\ 3\overline{)315} \\ \underline{300} \\ 15 \end{array} \qquad\qquad \begin{array}{r} 5 \\ 3\overline{)15} \\ \underline{15} \end{array}$$

In each solution the object was the separation or division of 315 into three equal groups. In solutions *b* and *c* the use of

zero is obvious. Since solution *b* was an outgrowth of solution *a* (child was told to write with numbers what he had written with other symbols), that solution is an integral part of the explanation.

Multiplication and Division with Tens

The background work in multiplication with tens is practiced incidentally in the oral arithmetic period long before systematic instruction is undertaken. Some of this experience comes as a result of adding tens, such as 8 tens and 9 tens. In this case the sum 17 tens is just another way of saying 17×10. The first systematic instruction in multiplication with a two-digit multiplier can be introduced easily through the use of problems involving 10×20 or 10×30. If this is done in an oral exercise, the children will not even realize that anything new is being undertaken (see Chapter 14), for they will already have been doing multiplication examples, such as 9×20, 9×30, and the like. The child's number record of 10×20 will usually take this form:

$$\begin{array}{r} 20 \\ \times\ 10 \\ \hline 200 \end{array}$$

In discussing solutions of this type, the question should be raised, "What is the size of the product, or what results when tens are multiplied by ten?" In order that children may understand the answer, each should demonstrate through addition or through the use of tens blocks that tens multiplied by tens result in hundreds or hundreds and thousands. A procedure designed specifically for development of this generalization may be delayed until after children have had experience with problems involving the multiplication of tens and ones by tens and ones.

A description of the procedures followed by a class in their study of this phase of arithmetic is given in the fourth-grade

illustrative lessons in Chapter 14 (page 375). In addition, the following steps may be of value in clarifying the process: (*a*) Writing a zero in the second partial product in order to emphasize the fact that the first recorded number is a ten and not a one. (*b*) Multiplying first by the left-hand digit of the multiplier. These steps are illustrated here:

(*a*)		(*b*)	
	23		23
	14		14
	92		230
	230		92
	322		322

Division by tens, like the multiplication of tens, should involve first only the use of exact tens (no ones included). For the introductory problems, measurement rather than partition should be used. The division of 60 by 20, using both types of problems, will illustrate the reason for this last recommendation. Oral problems using these division examples may be quite advantageous in clarifying the new procedure. When the solution is written with numbers, children usually notice that the quotient is the same as though ones had been divided. At least, this fact should be noted. The division statement $60 \div 20$ really asks how many 2 tens there are in 6 tens. Since children have already learned that tens are added and multiplied just as ones are, it should not be unreasonable to assume that in this case the same generalization can be seen. The division, then, is very nearly the same as that of ones.

When division by tens and ones is undertaken, the idea of approximation should be put to use. When confronted with an example such as $31\overline{)1726}$, the pupil should think how many 3 tens or 30's, not how many 31's. Changing the divisor into even tens is what a pupil actually does when he uses the device of covering with his finger the second figure of the divisor.

It should be noted that the divisor used as an example was approximately 3 tens. The task of estimating quotients is, of

course, more difficult if the divisor contains 4, 5, or 6 ones. Use of the indicated tens (or hundreds) is advocated if the second figure is 5 or less — e.g., in the example 846 ÷ 22, the divisor 22 is to be considered as 20. The tens should be increased by one if the second figure is 6 or larger. In the example 846 ÷ 26, the divisor 26 is to be considered as 30. Even if these procedures are followed, the trial quotient in division by tens will be the true quotient only about two-thirds of the time.

Since most division problems are of the type involving different collections (tens and ones, or hundreds, tens, and ones) some emphasis should be given to this matter of estimating quotients. Some teachers have successfully introduced the work through oral exercises. A problem which involves the division among members of the class of an amount of material, such as paper, is a good one to use. Suppose, for example, that there are 32 members in the class and that there are 150 sheets of paper to be distributed. The teacher says, "About how many sheets shall I give each of you?" After consideration of such a problem, the assignment for the class should be something like the following: "Here are a number of examples in which you must divide by tens and ones. Work the examples any way you can, but try to find a good scheme for handling such problems. Before you accept your scheme as the best, be sure you try to find at least one other scheme." The object of the work then becomes the finding of the best way of dealing with divisors that are made up of tens and ones. The same procedures and principles operate in the discussion of the pupils' solutions as in the beginning lessons on multiplication and division.

Since the trial quotient is more often the true quotient when large tens are used for divisors than when small tens are used, most of the introductory problems should have divisors above 50. The usual intensive-study period should follow the period devoted to getting an understanding of the process.

All modern texts have division examples and problems

arranged in the order of the difficulties involved. Whether or not the most obvious trial quotient is the true quotient is one of the main factors used in determining difficulty. The matter of zero in the quotient is another factor in determining difficulty. The procedures outlined above, in the discussion of division of hundreds and in the preceding paragraphs, plus a list of examples and problems, should be sufficient to guide the teacher in directing this phase of division work.

In an attempt to give opportunities for better understanding and for a better appreciation of the best methods of division, the teacher may suggest that division problems with tens and hundreds be done occasionally without the use of the division process. If children are entirely stumped by such a suggestion, careful consideration should be given to the type of problem and to the experiences children have had. For example, if a problem involving the number of 8-pound sacks that a boy can fill from a 200-pound barrel cannot be solved without using the division process, there is little likelihood that the pupils will get maximum profit from systematic instruction in that process. Serial subtraction, mediating (halving), and unwritten mental division of the larger number by separating it into its smaller parts are the methods most often used by children. The last method is, of course, very close to the division process. Some teachers have also suggested division on the abacus, where a form of serial subtraction is performed. The example below shows another procedure that has proved beneficial in gaining an understanding of the division process:

$$\begin{array}{r} 100 \\ 31\overline{)3761} \\ 3100 \\ \hline 651 \end{array} \qquad \begin{array}{r} 20 \\ 31\overline{)651} \\ 620 \\ \hline 31 \end{array} \qquad \begin{array}{r} 1 \\ 31\overline{)31} \\ 31 \\ \hline \end{array} \qquad 100 + 20 + 1 = 121$$

In multiplication where a zero is found in the multiplier, children often find it convenient to write a row of zeros in the partial product. In this way they keep numbers in their proper

places Although entirely unnecessary for and even detrimental to efficiency, this use of zero is legitimate and should be permitted just as writing of the carried number is allowed.

The major steps in the teaching of multiplication and division have been discussed in this chapter. Most of these steps will be learned by the children while they are in the latter part of the third grade, in the fourth grade, and in the early part of the fifth grade. A definite maintenance program, however, is needed to keep the skills on a high level of efficiency just as is true for addition and subtraction. In the case of multiplication and division, the maintenance program is particularly essential, because addition and subtraction occur more frequently in the everyday work of the pupils. Review or inventory tests, usually given at the beginning of work in the fifth, sixth, seventh, and eighth grades, should be the key as well as the initial step in this maintenance program. Since the principles governing such a program are practically the same as those outlined for the maintenance of skill in addition and subtraction, the reader is referred to page 163.

STUDY QUESTIONS

1. What advantage is there in beginning the teaching of multiplication with the 5's? (*1*) The 2's, 3's, and 4's, being smaller, will therefore be much easier to master. (*2*) The 5's rhyme and therefore are learned more easily. (*3*) The children know much about 5's and therefore have little to learn. (*4*) N.

2 The 1's in multiplication are difficult for some children. Why? (*1*) Because the product and multiplicand do not have the same relationship as in the 2's or 3's. (*2*) Because there really isn't any such thing as multiplying when 1 is involved. (*3*) There is no need for knowing the 1's. (*4*) N.

3. How many basic division facts are there (exclusive of the zero facts)? (*1*) 72. (*2*) 81. (*3*) 90.

4. What form of division, long or short, is used in current textbooks in teaching the basic facts? (*1*) Long. (*2*) Short. (*3*) Both.

5. Which of the two kinds of division problems, measurement or partition, should be used in introducing division? (*1*) Measurement. (*2*) Partition. (*3*) It does not matter.

6. Is the division of 32 by 4 a case of division of tens? (*1*) Yes. (*2*) No.

7. When will the child first have a need for multiplying 1's? (*1*) When learning the division of tens. (*2*) When learning to multiply by ten (*3*) When learning to multiply tens. (*4*) N.

8. Which of these is classified as an easy multiplication fact? (*1*) $4 \times 7 = 28$. (*2*) $8 \times 4 = 32$. (*3*) $6 \times 6 = 36$. (*4*) N.

9. Some teachers pay little attention to the order of difficulty in the presentation of the multiplication facts What argument do they use for not postponing the most difficult facts until last? (*1*) All the time should be devoted to the difficult. The easy facts will then be learned incidentally. (*2*) All facts have to be learned in a relatively short time and therefore the child is little better equipped mentally because of postponement. (*3*) Relationships are more important than difficulty and both cannot be emphasized. (*4*) N.

10. What advantage is there in using tables in learning the multiplication facts? (*1*) The making of assignments is made very easy. (*2*) Little space is needed to present the facts. (*3*) Complete check of a child's accomplishment can be obtained with a minimum of effort. (*4*) N.

11. Why is a partition problem better than a measurement problem for introducing the division of tens? (*1*) Because in a measurement problem the dividend must usually be changed to ones. (*2*) Because in partition you think in terms of ten. (*3*) Because partition is true division. (*4*) N.

12. Should children eventually be taught to write all remainders in division as fractions? (*1*) Yes. (*2*) No.

13. In addition to carrying, what new thing does the pupil have to master in learning the multiplication of tens? (*1*) Proper placement of partial products. (*2*) Multiplying zeros. (*3*) Addition of partial products. (*4*) N.

14. In determining the quotient why is the trial or estimated quotient more often incorrect when the divisor is a twenty number (21, 22, etc.) than when the divisor is an eighty number (81, 82, 83, etc.)? (*1*) Because the number of ones not considered with 20's is usually larger. (*2*) Because a twenty number is more often used as a divisor than an eighty number. (*3*) Because the quotient when 2 is a divisor is larger than when 8 is a divisor. (*4*) N.

15. What is the purpose of having fifth-grade pupils sometimes solve a division situation (e g , 600 ÷ 40) by methods other than division? (*1*) To give the pupils a better foundation for division with tens (*2*) To demonstrate that division is a short process. (*3*) To show the relationship between division and subtraction. (*4*) N.

16. Why is the use of the form $5\overline{)15}$ for partition division misleading? (*1*) Because partition problems do not require division. (*2*) Because the expression, "how many in," does not apply to partition problems (*3*) Because partition can be indicated by fractions only. (*4*) N.

17. Pupils who have used the multiplication tables in learning the facts sometimes resort to going back; for example, when confronted with 8 × 8 go back to 3 × 8, 4 × 8, and so on. Is this fact good argument for elimination of the tables? (*1*) Yes. (*2*) No

18. It is generally conceded that long division should be taught before short division. Is there any valid argument for teaching short division? (*1*) Yes. (*2*) No.

19 Why should most of the pupils' early experiences with multiplication facts be with the column rather than the equation form? (*1*) Because the column form is the one that can be developed more easily from a problem. (*2*) Because that is

the form in which all of their important uses of multiplication will appear. (3) Because the equation form is too much like the table form. (4) N.

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7

Problems and Problem-Solving

Purposes of Problem-Solving

The term *problem* in arithmetic means a quantitative situation described in words in which a definite question is raised, but for which the arithmetical operation is not indicated. While this is an admittedly narrow and arbitrary definition, it is, as Morton has stated, a useful one. It makes for easy distinction between problems and examples, the latter being number situations for which the arithmetical operation to be performed is indicated, as 6 × 24.

Skill in the solution of problems is considered by many as the major objective of all arithmetical work. This point of view is well expressed by Morton when he states that the purpose of the work in the four fundamental operations is problem-solving [1] While the ability to solve problems is an important aspect of arithmetic, it is doubtful whether the skill should be set up as a separate objective, the end of all instruction in the fundamental processes. Rather than being considered as some-

[1] R. L. Morton, *Teaching Arithmetic in the Elementary School* (New York: Silver-Burdett Company, 1937–38), I, 346, G. M. Wilson and others, *Teaching the New Arithmetic* (New York: McGraw-Hill Book Company, 1939), pp. 295–317.

thing separate from other phases of arithmetic, problem-solving should be held an integral part of the total program. For example, in teaching the multiplication process, problems were used in illustrating the procedure, in showing the significance of the multiplication process after it was mastered, and in review and test exercises of the multiplication process. Every modern textbook makes use of problems in these ways. The six common functions of problems, including those just mentioned, are listed and briefly discussed in the paragraphs that follow.

1. The use of problems to illustrate new concepts and new processes is a major characteristic of the instructional procedure advocated in this book. The problem in these cases furnishes a significant setting for the introduction of something new; it provides experiences to aid the child in grasping the mathematical fact or procedure that is being taught. Since so many new facts and procedures are introduced in the first four grades, problems are frequently used for illustrative purposes.

2. To show that facts and processes are useful to the students, texts include problems using those facts and processes. Such problems are usually presented after the student has had a reasonable opportunity to master the procedure called for in the solution of the problem, although they may appear as an aid to acquiring mastery.

3. The function of problems in review exercises and in tests is so obvious that little comment seems necessary. In the review exercises, problems are especially useful in illustrating any re-teaching. The importance of problems in tests is shown by the fact that at least one-fourth or more of the total testing time in the arithmetic sections of achievement tests is devoted to problems.

4. Problems are frequently used to unify a class, to create class spirit. For example, during periods of intensive study on a series of facts or on a process, each member of a class is rightly

working on his particular weakness, and therefore all work is of an individual nature. The use of problems at such times serves to bring the class back together as a unit. Problems furnish a much better basis for discussion than do examples, and every member of the class feels that a new start is being made in which everyone is at the same place. These problems may require only verbal solutions or they may require the use of pencil and paper.

5. Problems are frequently used in instruction to provide practice in the processes that are being taught. Although problems are decidedly inferior to some other means of providing practice, their use is recommended as offering variety. Furthermore, practice problems give significance to the process being studied.

6. Problems are used in arithmetic texts to teach pupils how to solve the situations they will meet in life outside the arithmetic classes. Finding areas and averages, making change, figuring the cost of installment buying, and the like, are good examples of this type of problem. Such problems are used much more frequently in the fifth, sixth, seventh, and eighth grades than in the first four grades.

The six functions of problems listed in the preceding paragraphs indicate that problem-solving is an integral part of the instructional program. Each clearly indicates, as well, one of the purposes of problem-solving. Only the sixth use may be easily classified as something separate, not essential in teaching the fundamental processes of arithmetic. Teaching pupils to solve the problems of life may of course be considered the main goal toward which the other uses of problems point. If, however, problem-solving is considered only as getting the answer to the question in a verbal problem, it is a mistake to assume that all instruction in arithmetic is directed toward this end. There are other uses of arithmetic than those shown in the

solution of verbal problems. (See under "Proposed Purposes of Arithmetic," in Chapter 2.)

The use of problems to illustrate new facts and processes is not unique with the program advocated by this book.[1] In fact, almost every modern textbook introduces new phases of arithmetic by means of problems. But as indicated by Morton, the primary purpose of using problems in the introduction is to enable pupils to "see a need for acquiring skill in a particular process."[2] That, of course, is not the primary purpose of introductory problems as outlined in this chapter. While children may eventually see a need for acquiring skill in the process as a result of work with problems, the first use of problems is to furnish a situation in which children may gain an understanding of the process and to see one of its functions. The underlying causes for the different roles assigned to problems by current practice and by the writer are further developed in the section on "Selection of Curriculum Materials," especially "Logical Analysis Method of Selection," in Chapter 11.

The uses of problems show that problem-solving is an essential part of the total program and not something only related to the other parts. Thinking of problem-solving as something separate often results in misplaced emphasis. For example, when children appear to be weak in problem-solving, improvement is frequently sought by devoting a greater portion of the total arithmetic time to the solution of problems. If problem-solving were as distinct a phase of arithmetic as, for example, column addition, then the extra time given to working problems might result in marked improvement. Weakness in problem-solving can probably be most effectively combatted through proper emphasis on the uses for which problems in arithmetic are intended, and through special attention to procedures which

[1] H. G. Wheat, *The Psychology and Teaching of Arithmetic* (Boston: D. C. Heath and Company, 1937), pp. 218 ff.

[2] R. L. Morton, *Teaching Arithmetic in the Elementary School*, II, 73.

pupils have found helpful in the solution of problems. Procedures which make for the development of the ability to solve problems are discussed in the section that follows.

Development Program

An essential part of a program for developing the pupil's ability to solve problems is recognition of the true purposes of problems in the instructional program. During the early years problems are employed primarily to illustrate the mathematical facts and processes which the child is to learn, to show him the uses of those facts and processes and give him practice with them, and to keep the class working as a unit. Recognition of such purposes of problem-solving should result in the selection of problems closely related to child experience. The child should be thoroughly familiar with the details and situation of a problem if it is to illustrate or clarify a fact or procedure. No aspect of the problem should present special difficulty, and the pupil should have no trouble in working out the answer. All effort should be centered on the finding of a new fact or a better procedure. Suppose, for example, that the child who does not know that 4 and 4 equal 8 is confronted with this problem situation: "Jack put 4 valentines in the box and Mary put in 4. How many valentines did the two children put in the box?" He should know that he can count the valentines to find how many in all. Thus, he finds out that 4 and 4 equal 8 and is therefore in a position to accept that finding as a fact. He has demonstrated to himself through the setting provided by the problem that 4 and 4 equal 8. On the other hand, suppose that the child cannot count to get the answer to the question. The problem is not, then, a means of illustrating the fact that 4 and 4 equal 8. In such a situation the real task of the child becomes that of finding the answer to a question, a task which under the conditions described would be either very difficult for the child or almost totally lacking in meaning. The other purposes for

which problems are used in the early grades have already been discussed. Acceptance of the purposes proposed for the use of problems will result in a way of using them which is markedly different from the way in which they are now commonly used in the first four grades. It has already been stated that these same purposes are applicable in the upper grades, but there, in addition, the solution of real-life problems becomes a major objective. The use of problems for test purposes also becomes more important in the upper grades.

The reader's attention is called to the fact that recognition of the purposes for which problems should be used, or even of the validity of the purposes proposed in this chapter, is seldom reflected in the proposals generally made for the improvement of problem-solving in arithmetic. For an example of a different view of the role of problems and how to improve problem-solving, the reader should consult Wilson's statement.[1]

Another essential feature of a program for developing the pupils' ability to solve problems is an emphasis on the number system. The chief economy-making features of the system should be stressed — the fact that tens and hundreds are added or divided just as ones are, that all numbers larger than ten are collections, that the fundamental processes are short ways of doing group counting, and the like. In this part of the program for development of problem-solving ability, emphasis should be placed upon the nature of the processes and their interrelationships. Since numbers are used in the solution of problems, it may be concluded that problem-solving will be facilitated by a better understanding of the meaning of numbers and of the various processes with numbers. It should be noted that this part of the program does not involve direct work with problems.

The development of the ability to solve problems will be aided by the liberal use of objects and symbols (concrete and semi-

[1] G. M Wilson and others, *Teaching the New Arithmetic*, chaps. 24, 25

concrete) in showing the meaning of numbers and processes. Consider, for example, such a problem as "How many are three books and four books?" The teacher might suggest that the child represent three books by showing three fingers on one hand, and four books by showing four fingers on the other hand. The child is next asked how many fingers in all, and then how many books in all. Marks, squares, or other semi-concrete ways of representation may be used instead of fingers. To help the child see that the two amounts are combined or added into a single group, a circle is sometimes drawn around the representation of both quantities. This use of objects, symbols, and devices should include the use of the actual objects referred to in the problem. The procedures just described are most helpful in Grades One through Four, and are of value primarily in laying a foundation for problem-solving. Some textbooks planned for Grades Two and Three make extensive use of semi-concrete representation of the facts stated in problems.

Oral arithmetic should also be included in any program designed to develop ability in solving. When the procedure is carefully examined, it is not difficult to see how problem-solving ability is furthered by oral practice that does not make use of pencil and paper. In written solutions the act of writing takes both time and attention and may interfere with the pupil's thinking. In the oral solution there is no such interference. The written solution provides the pupil with a record of his thoughts and thus relieves the mind of remembering The lack of record in the oral solution forces him to give closer attention to the processes and steps he uses, and to concentrate on the most important. For purposes of illustration, suppose that the problem involves the addition of 38 and 43. In the effective oral solution the pupil takes two of the three ones in 43, combines them with the 38, and then adds 40 and 41, thus going directly to the process of making tens and ones. In the written solution he can easily get the correct answer without realizing that he has

rearranged the tens and ones of the two numbers into a single equivalent number of tens and ones. Oral arithmetic, then, tends to emphasize significant aspects of the number system. Oral solutions to problems not only lead the pupil to use such significant aspects of number processes as that described above, but also give him practice in rounding numbers and in arriving at approximate answers.

The use of approximate answers and rounded numbers is another procedure which helps toward the mastery of problem-solving. In many quantitative situations an exact answer is not required in order to get usable ideas of the quantities involved. Consider, for example, a problem such as the following: "How does city A, population 81,309, compare in size with city B, population 319,845?" Here, as in most situations involving size of population, the round number 80,000 is probably a far better figure to use than the exact number. Where a rounded number serves the purpose, it makes for thinking that is simpler and usually just as meaningful. The exact figure, however, would be more suitable than the approximate amount represented by the rounded number in answering accurately a question like "How much has the city grown since 1940?"

Children who learn to use approximation in solving problems either orally or through the use of pencil and paper have confidence in their ability to solve problems when the exact answer is demanded. That approximation of answers is beneficial to facility in problem-solving has been demonstrated in several studies. By devoting about one-fifth of their allotted arithmetic time for one semester to approximation, seventh- and eighth-grade children in one school improved more than one and one-half years in problem-solving ability.[1]

[1] Wallace Wood, *A Study of the Growth of Pupils in Arithmetic When Answer Approximation Is Emphasized* (Unpublished Master's Thesis, State University of Iowa, 1941).

Closely related to approximation is the procedure known as estimating answers. In fact, some people make no distinction between the two processes. In general arithmetical practice, however, answer estimation has become definitely subordinated to problem-solving. In the most common procedure, an estimate of the answer is made before the exact computation is performed. After the computation, the exact answer and the estimated answer are compared. Obviously, this is a good scheme to use in order to avoid gross errors, and the approximation used in connection with estimation of answers will undoubtedly yield some of the advantages claimed for approximation in the preceding paragraph.

The use of diagrams and other means of representation, as described in the presentation of the methods of instruction (Chapter 2) and in the introduction to the processes with whole numbers and fractions (pages 125 ff. and 271 ff), is an important procedure in the development of problem-solving. The drawing of a diagram or other illustration to show what is told in the problem with words requires a thorough understanding of the problem. A prerequisite to such understanding is careful reading. Often major misconceptions about a problem are not recognized until the child starts to represent the situation graphically. For example, in representing the situation in a problem which stated that Jack carried two plates at a time and made seven trips, a child first drew two groups of seven plates each. The teacher merely said, "Name the trips: trip one, trip two, and so on." The child himself immediately discovered his error. The diagram or other illustration is a test of understanding. It is especially important in the early grades where practically all problems can be illustrated. In the upper grades there are problems that cannot be easily diagramed. For such problems a type of analytical approach is suggested. (See the method of Unitary Analysis as described in Chapter 10.) It should be recognized that the problem-solving procedures which

emphasize graphical representation have the very serious limitation of requiring a great deal of time.

As has often been demonstrated, problem-solving will benefit from practice in careful reading. Some of the better exercises to use in such an improvement program are reading to find: (1) what facts are given; (2) what is to be done; (3) an approximate answer; and (4) what process is to be used. In training a pupil, it is best to practice only one of these exercises with each problem employed;[1] that is, one problem should be scrutinized for finding the facts, another for finding what is to be done, and so on. In this way the child's attention is centered on the one exercise. If all the exercises were practiced on one problem, the child would soon consider such careful reading as a routine piece of busy-work. Instead of reading carefully for the third and fourth items, the child would try to answer them without reading at all. Forcing a child to go through all four exercises on every problem does not appear to be a good procedure, if improvement in reading is the goal. But, as indicated above, the use of the various items with different problems is a constructive training device, because at least one careful reading of each problem will be necessary.

In the efficient solution of any problem three of the exercises suggested for improvement of reading seem fundamental. Before the pupil can solve a problem he must: (1) find out what facts are given; (2) find out what question is asked; and (3) decide what operations must be performed in order to answer the question satisfactorily. It seems logical, therefore, to conclude that a pupil's ability to solve problems will increase with training in the systematic application of these three items to the solution of problems. Many textbooks for children include exercises of this type in their problem-solving program. Frequently the exercises are elaborated to include more approaches, and sometimes Item 1 is omitted. The directions for the pupil

[1] J W A. Young, *The Teaching of Mathematics in the Elementary and the Secondary School* (New York: Longmans, Green and Company, 1907), p. 208.

are usually of this type: "Answer each of these questions about the problem and then solve the problem." That this formal-analysis method of solving problems has value is attested by the fact that so many textbooks recommend its use. Although the steps in the formal-analysis method of problem-solving are sound, many teachers have been disappointed by their experience with the procedure. Pupils fail to do the steps in order; they take so much time to produce evidence that they have used each step that they lose interest; and so much attention is focused on the first steps in the procedure that the later steps are often carelessly done. Perhaps the difficulty experienced in teaching the formal-analysis method is due primarily to the mechanics of the procedure used in getting pupils to carry out the steps. On the other hand, it must be admitted that few adults when they solve a problem, consciously follow in sequence the steps in the formal-analysis method. Frequently, the adult uses a sort of intermingling of steps and has in mind no distinct and clearly recognizable progression. To ask children, therefore, to apply the steps in a fixed sequence in the solution of problems may not be in harmony with the best adult practice.

In Chapter 5 (Addition and Subtraction), the use of cues in teaching children how to solve problems was discussed briefly. Cues are not recommended because it is believed that attention given to the cue takes the mind of the pupil away from the main task before him and will in the long run be detrimental to his ability to solve problems. For other procedures which are claimed to make for improved problem-solving, and for a more complete discussion of some of the steps listed, the reader is referred to such books as those written by Morton [1] and by Brueckner.[2] The topic of two-step problems is also adequately treated in such references.

[1] R. L. Morton, *Teaching Arithmetic in the Elementary School*, II, 454–93.

[2] L. J. Brueckner, *Diagnostic and Remedial Teaching in Arithmetic* (New York: John C. Winston Company, 1930), pp. 259–96. See also Josephine MacLatchy, "Variety in Problem-Solving," *Education*, *61*, 453 (April 1941).

Before leaving the present subject, the characteristics of good problems should be considered. Most authorities are agreed that problems should be real; that is, they should deal with things and situations that are within the experience of the children. This characteristic is most important in problems which are used in the early grades for purposes of illustration. Obviously, such problems must of necessity deal with experiences familiar to the child; otherwise they will not make clear to him the situations which they are intended to illustrate. Although problems which deal with the interests and experiences of children are not difficult to find, the teacher must often do some careful planning in order to see that suitable problems arise or are at hand at the appropriate times.

The experiences of children, however, are not so nearly identical in character that problems which arise in the classroom will be real to all the children in the class. In other words, a problem is not necessarily real just because its setting is in the classroom. A problem is real when the child sees that its solution is essential to the achievement of some larger end toward which he is working. It is real when the child sees a need for solving it. This need does not have to arise in the child's own situation. He may identify himself with some other person's needs and can adopt problems from imagined situations. Textbook writers have made extensive use of the last two conditions, and have been very skillful in creating make-believe situations out of which real problems can arise. In their efforts, however, to make imaginary situations plausible, they have included factors which have no direct bearing on the questions raised in problems. Consider, for example, the following:

John mowed $\frac{1}{4}$ of the lawn on Friday. Then on Saturday Dick and Bill came over and helped him finish the job. While Dick and Bill each mowed $\frac{1}{4}$, John mowed the other $\frac{1}{4}$. How much of the lawn did John mow in both days?

A similar problem-situation is presented in a fifth-grade textbook, but in more elaborate detail. John wanted to go to a

ball game on Saturday afternoon, but his father had said that the lawn had to be mowed first. The two other boys wanted John to go to the game, too, so they came to his aid. Without question this is a clever setting, but notice that the ball game really has nothing to do with adding $\frac{1}{4}$ and $\frac{1}{4}$

Besides reality, other general factors of problems that merit some consideration are interest, language, and applicability to the arithmetic that is being taught. All of these factors are discussed at length in other books.[1] It should be obvious that problems which are interesting, which use words that the children can read and are illustrative of the type of arithmetic being taught, are superior to problems which do not possess these characteristics.

A Critical Analysis of Problems

In the ordinary verbal problem a question is asked in terms of a word description of a quantitative situation. The statement that most satisfactorily answers the question can be secured by the use of one or more of the fundamental operations. As an illustration, consider the following problem: "We have twelve chairs, but there are sixteen visitors. We must have a chair for each guest. How many more chairs do we need?" Here two numerical terms, twelve and sixteen, are used in pointing out the inequality between the number of available chairs and the number of visitors. The size of this inequality is made much clearer by subtracting twelve from sixteen. Obviously, it is easier to get a clear idea of the exact difference from picturing this one quantity of four chairs than it is from any picturing of twelve chairs and sixteen persons. It should be observed that the problem in word form presents all the conditions, all the facts. The solution results in simplification of the situation. Before the problem was solved, the best statement of the situation was, "We have twelve chairs, but there are sixteen visitors.

[1] For example, Morton, *op cit.*, I, 348–64

We need more chairs." After the problem was solved, the best statement of the situation was, "We have twelve chairs, but there are sixteen visitors. We need four more chairs."

The accepted number solution, however, is not always the simplest solution for the child. Since a number solution deals with abstract numbers, it may not be understood; therefore, the one-number answer, instead of clarifying the situation, may not mean anything to the child. As has been stated on so many occasions, the child must have the experience which will enable him to see that the number solution is a simplification. In the case of the illustrative problem above, the child may need to draw twelve chairs with a person seated on each one and then have enough people standing to make sixteen. This is merely saying that the semi-concrete representation is usually understood better than the abstract. It is through concrete and semi-concrete representation of the conditions of the problem that a child gains confidence in the number representation of the conditions.

Ordinarily, the number solution of a problem is not thought of as a short way of representing the same conditions that are stated with the words, plus, of course, a rearrangement of the numbers of the problem. As an illustration of this point, the two ways of representing the conditions in the "visitors and chairs" problem are shown below.

In reality	*In numbers*	
(*a*) Sixteen persons are in one place, twelve chairs are in another.	(*a*) Write numbers	16 12
(*b*) Twelve persons go from group of sixteen to occupy 12 chairs.	(*b*) Subtract	16 − 12
(*c*) Number of persons left in original group are counted to find the number of chairs required.	(*c*) Write remainder	16 − 12 4

As stated above, this rearrangement should result in a simplification of the conditions of the problem. The solution to a problem cannot present any new facts or conditions. It can only state the given conditions in more easily understood form. Of course, the new form is more easily understood only with reference to the question raised in the problem. Looking upon solutions to word problems as merely a simpler way of stating the conditions of the problem will permit the acceptance of pictures, diagrams, fragmentary statements, and other procedures as solutions right along with the commonly accepted number solution. Viewed in this light, problem-solving becomes quite a different matter from answer-getting. Furthermore, acceptance of this type of problem-solving will make it easy for the teacher to provide for individual differences. For an illustration see lesson 3, Chapter 14.

The solving of verbal problems in arithmetic is often considered good training in the steps in thinking as outlined by Dewey[1] Before considering this statement it may be well to state briefly these steps. Reflective thought arises from a state of doubt or mental difficulty. In attempting to clarify the difficulty five phases of thought are recognizable. The first phase, that of suggestion, consists of two or more contradictory and rather simple things that might be done to solve the difficulty; the second phase, that of intellectualization, is a general sizing-up of the total situation, a sort of over-all observation of all conditions; the third phase, the guiding hypothesis, is a possible course of action, or explanation; the fourth phase, that of reasoning, consists of a full development of the idea suggested by the hypothesis; and the fifth phase, of testing the hypothesis by action. In the solution of a verbal problem the answer to a clearly stated question is found. The question of a verbal problem creates the difficulty and defines the problem for

[1] John Dewey, *How We Think* (Boston: D. C. Heath and Company, 1933), pp. 106–15.

thought. There is, therefore, little opportunity for intellectualization. Furthermore, the selection of a hypothesis in the solution of verbal problems almost automatically decides the arithmetical process, and the reasoning phase then practically disappears. The fifth phase — testing the hypothesis — is not very useful in the solution of a verbal problem. From these statements it can be seen that the solution of verbal problems in arithmetic is not the best illustration of the steps in reflective thought. There are other situations in arithmetic which better exemplify the process of reflective thinking than does problem-solving. An example is the examination of various solutions for the purpose of finding the best solution. Perhaps a still better illustration is the thinking which a pupil does when he realizes that his answer is absurd and tries to find out what is wrong.

The Social Content of Problems

At present many studies in the field of arithmetic are concerned with what has become known as the social side of arithmetic. The writers of this literature insist that the social scene portrayed by problems and other descriptive material in arithmetic texts be true to life. The requirement that problems be true to life is somewhat related to the matter of reality in problems, discussed in a previous section, where the importance of using real problems was pointed out. It will be recalled that reality was emphasized primarily as a factor in insuring comprehension on the part of the children. Other and quite different motives for using real problems, however, are also frequently advanced. For example, Wilson[1] insists that real-life problems be used, not only because they will be more easily understood, but because they will be of more interest to the child. He argues further that through the use of life problems arithmetic

[1] G. M. Wilson, and others, *Teaching the New Arithmetic*, pp. 300, 301, 303, 304.

becomes subservient to the child. Careful study of Wilson's proposal for teaching problem-solving reveals that the teaching of economics or social customs assumes a major role in the problem-solving program. Many other writers have advanced the idea that problems in arithmetic should be social in character. The committee on arithmetic of the National Council of Teachers of Mathematics refers to the social phases of arithmetic. Perhaps the most insistent campaigner for using social situations in the teaching of arithmetic is Brueckner.[1]

The value to arithmetical development of problems that deal with social situations has not been investigated experimentally and must, therefore, be decided on a subjective basis. Suppose, for example, that the addition of 180 and 235 is the arithmetical operation required by two problems. One problem uses the number of jewels in the crowns of a fairy king and queen while the other uses the number of minutes a week that the children devote to two subject areas. Most people will say that the use of minutes is the better problem, not because the addition will be easier or better understood, but because the problem is more sensible than the other, more true to life. It should be recognized that the use of social situations in problems does not necessarily make the arithmetical operations required for the solution of the problems more meaningful. Consider, for example, the following: "The fairy left four jewels under each of the three pillows. How many jewels did she leave?" and "The favors Betty bought for the three guests cost four cents each. How much did she pay for all?" For which problem could the pupil more easily show with a diagram what is told in words? Which problem would be easier for the pupil to solve? Which problem would better illustrate that three fours equal twelve? In answering these questions most teachers would

[1] For a brief view of Brueckner's position, the reader should consult Chapter VII of the *Sixteenth Yearbook* of the National Council of Teachers of Mathematics

probably say "Either one," or "The first." A good principle to follow, then, is to use the social situation in arithmetic when that situation has a contribution to make to arithmetic, and not just because it is a social situation.[1]

STUDY QUESTIONS

1. For what reason is "skill in problem-solving" not a good goal or objective for all work in arithmetic? *(1)* Examples are almost as important as problems. *(2)* Problem solutions do not comprise nearly all the important uses of number. *(3)* Because problems are always recorded and involve reading. Some people cannot read and yet can use numbers. *(4)* N.

2. Which of the major uses of problems is most important in the third grade? *(1)* To unify the class. *(2)* For review and test exercises. *(3)* To provide practice on basic facts and processes. *(4)* N.

3. Why is problem-solving so often considered as something separate from the rest of arithmetic? *(1)* Because the purposes of problems in the instructional program are not recognized. *(2)* Because problems sections are presented in words, not figures. *(3)* Because problems are more difficult than examples. *(4)* N.

4. When problems are used to introduce a new fact or process, should the child be able to answer the question of the problem? *(1)* Yes. *(2)* No.

5. Oral arithmetic contributes to improvement in problem-solving. What factor might account for the improvement? *(1)* Oral arithmetic is faster and therefore more problems can be considered. *(2)* Oral arithmetic tends to focus attention on the more significant aspects of both the problem and the solution. *(3)* The confusion resulting from writing is avoided. *(4)* N.

[1] For an extensive treatment of this problem, consult H. G. Wheat, "The Fallacy of Social Arithmetic," *Mathematics Teacher* (January, 1946), pp. 27–34.

6. Having the child make a diagram to show what the problem tells with words is commendable for several reasons. What is probably the most important? (*1*) It requires the child to visualize the situation. (*2*) It requires thorough reading (understanding) of the problem (*3*) It avoids abstract numbers. (*4*) N.

7. What is the most serious limitation of the use of diagrams in the solution of problems? (*1*) Some children cannot draw. (*2*) The drawings being semi-concrete representation make unnecessary the use of numbers. (*3*) The drawing requires too much time. (*4*) N.

8. When trying to develop careful reading ability why should only one exercise be used with each problem? (*1*) Because there is usually only one aspect of a problem that warrants attention. (*2*) Because extensive not intensive reading is needed most in arithmetic (*3*) Because the solution of the problem should not be delayed by use of practice exercises. (*4*) N.

9. What is the main reason for opposing the use of cues in the solution of problems? (*1*) Cues just require extra time. (*2*) Cues direct the attention of the pupils away from an over-all consideration of the problem. (*3*) The task of identifying the cues is often more difficult than the solution of the problem. (*4*) N.

10. For a problem to be real to a child, does it have to be one of his own or that of someone with whom he is acquainted — for example, a classmate? (*1*) Yes. (*2*) No.

11. In the solution of a verbal problem in arithmetic is much practice given in the complete act of thought as Dewey describes that act? (*1*) Yes. (*2*) No.

12 What is the major argument for the use of socially useful content in problems? (*1*) It makes the solution of the problems less difficult than otherwise. (*2*) Some socially useful material is taught at the same time that arithmetic is being taught. (*3*) Problems built on socially useful content are easier to diagram. (*4*) N.

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8

Weights and Measures

The Status of Measurement

The modern arithmetic text makes extensive use of measures. A glance at the problems section will reveal frequent occurrence of quarts, bushels, minutes, tons, dollars, and the like. If enumerating and the expression of "how many" by use of numbers are considered measures, then practically all problems involve some form of measurement. Thus, it seems that the teaching of arithmetic is dependent upon the use of measurement. However, important to arithmetic as measurement is, yet arithmetic is not dependent upon measurement. Careful examination of problems will show that the measures are used primarily to make the work with numbers appear more significant. In fact, if enumerating is not considered as a form of measurement, all operations can easily be taught without the use of measures. Many present-day texts actually do teach the fundamentals of addition, subtraction, multiplication, and division without the use of common measure. The role of measurement in the teaching of arithmetic, although important, is not, then, so essential as a glance at arithmetic problems would seem to indicate.

Before considering further the place of measurement in arithmetic, careful attention should be given to the major character-

istics of measures. In the discussion of number-concept building, several exercises (see examples 8 and 9, Chapter 4) were concerned with developing some of the fundamental ideas of measurement. In one of the exercises, the children showed, by reference to the height of tables and desks in the schoolroom, how high the tomato plants in their garden were In such situations an object near at hand is used as a substitute for the original object under consideration. The children used the near-at-hand tables and near-at-hand desks as measures of the height of their tomato plants. This simple use of an available object to measure a dimension of an object not immediately available illustrates one of the major characteristics of all measurement: namely, substitution. One object (the measure) is used to take the place of the real object. By means of the substitution of measures for the object or quantity under consideration, we are able to reproduce the magnitude of any quantity. It should be noted that this substitution involves comparison or matching, and therefore measurement is only as accurate as the measurer's ability to compare It should also be noted that the substitute (the measure) for the quantity being considered is usually less difficult to visualize, to manipulate, and so forth, than is the original quantity.

For clarification of the ideas just presented, consider four simple measurement experiences of children:

1. A child has built a house with twelve blocks and he wishes to duplicate the structure in another room. He knows that he needs a number of blocks, but does not know exactly how many. He therefore picks up some blocks at random and takes them to the other room. Then deciding that he does not have enough, he goes back and counts the number of blocks used in the original house. He then carries away enough more blocks to make up the twelve needed. Note the matching (comparison) of ordinal numbers to blocks which this child did in order to find how many were needed. The other obvious

way to duplicate the quantity of blocks would have been to match block with block. Certainly, the matching of number names with blocks was simpler than matching block with block. Note that in this measurement situation discrete objects were compared by counting, and therefore the usual errors caused by inability to compare do not occur. The reader should recall that counting is the only exact mathematical operation that man can perform on objects he perceives.

2. In a story Jerry found this statement, "The panther was five feet from his head to the tip of his tail." With a ruler Jerry marked five feet end-to-end on the floor. Jerry then said, "This is how long the panther was."

In this situation the length of the panther was duplicated by use of measures, and the child was thus able to get a good idea of this aspect of the panther's size, even though no panther was visible.

3. "I have a rabbit that weighs ten pounds," announced Bill one morning.

"Is that too heavy to carry?" asked Nancy

"I haven't tried," was Bill's answer

The teacher suggested that it was possible to answer Nancy's question without trying to carry the rabbit. After some discussion books were weighed. When a number of books equal to ten pounds was accumulated, several children lifted them.

Here again is an example of the use of measures to reproduce the magnitude of a quantity not immediately available. First the actual weight of the object (rabbit) had been compared (measured) with a standard measure. Then, the standard measure was again compared (made equal) to the actual weight of some other objects (books).

4. "Let's see how long it takes us to walk home," said Bill to his sister. "It's now four o'clock."

When the two got home it was four-twenty. Bill said, "It took us twenty minutes."

In this case an amount of time was measured by comparing with a standard time interval. This standard measure could

then be used in describing the time instead of such indefinite statements as "a long time" or "quite a while."

From the examples above it can be seen that measurement is basically a matter of comparison A defined or well-known series is compared (as in counting) with some quantity; or, more generally, some standard unit (gallon, foot, or the like) is compared with the quantity under consideration. The comparison of some quantity with a measure such as a foot, a gallon, or a ton is of course the chief use of measurement

From the counting of actual objects, a simple form of measuring, the procedure to follow in teaching children the basic ideas of measurement is not well established. In fact, so few writers have even considered this phase of teaching that almost no help is obtained from a careful study of the literature. Most programs as presented in children's textbooks just start the children in the use of standard measures without providing experiences which would give them an opportunity to see the reason for using those measures. In this work, as in teaching the fundamental processes of arithmetic, it seems to be taken for granted that because these standard measures are used in life, children ought to see a need for using them in school. To go from the numbering of objects to the use of standard measures is, without question, quite a step. Many experiences with various aspects of measurement should come between these two steps. It is recommended, therefore, that a long period of concept-building precede actual work with standard measures and systematic instruction in the use of such measures.

Concept-Building Program

Some idea of the type of exercises to use in a concept-building program has already been given in lessons 8, 9, and 12, in Chapter 4. The major purpose of these lessons was to develop the idea that a simple object near at hand can be used as a substi-

tute for the real object to be defined. It is through such exercises that a child learns the meaning of and the reason for measurement.

The lessons included in Chapter 4 dealt only briefly with measures of time, weight, and length. Each of these aspects of measurement has many ramifications In addition, such important phases of measurement as volume and capacity, area, value, angle, and temperature are to be developed. To give the child experiences that will make for an understanding of measures in these various aspects, the following procedures are suggested. Although intended for use in the primary grades, some of these procedures may be suitable for children in the intermediate grades.

1. Using liquid measures (cup, pint, gallon, and the like) in connection with such activities as cooking, filling the aquarium, and finding how much milk a cow gives.

2. Planning the school garden or a section of it on the room floor and later transferring an area of that size to the garden by the use of strings or other available measures.

3 Deciding whether or not a cage or a box will fit into a certain space

4. Making a frieze large enough to fit a specified area.

5 Finding whether or not the first-grade chicken house can be brought into the schoolroom for repairs.

6. Keeping a record of the time required for birds' eggs to hatch.

7. Having the pupil see how far he can walk in a minute. Letting the child count for a minute, the child himself doing the timing. (The distance walked may be measured in feet, but it is probably better to use the number of times around the room or the distance from the building to a tree.)

8. Weighing ingredients to go into a cake.

9. Marking on the blackboard the length of jumps made by different children, the distance jumped being transferred from the playground by means of a long stick.

10. Selecting objects on the playground or in the room that are as heavy as some special object, such as an ear of corn seen on a visit to a farm.

11. Marking on the wall the height of common farm animals.

The concept-building program should also provide experiences which will permit the child to become familiar with the most common geometric forms. The circle and the square are used so frequently in connection with such exercises as "Put as many marks in the circle as there are in the square" that no special attention need be given to them. The rectangle, triangle, hexagon, octagon, and parallel lines need special attention. The textbook with its diagrams usually presents adequate initial instruction concerning these geometric forms, but the child's understanding will be greater if he is given a chance to make some use, other than passing the book test, of his knowledge of the forms. The concept-building program can provide such uses.

During the time that the concept-building program is in progress, children will use many standard measures. In fact, some of the exercises, such as cake-making and the weighing of objects, require the use of standard measures. No attempt should be made at that time to explain the meaning or reason for using such measures. The major purpose of the program is to develop the idea that simple things (measures) near at hand can be used in describing the quantity under consideration. Another basic idea of measurement that is developed through these exercises is that measures are a means of reproducing in other places the quantity under consideration, as when the distance jumped in the school yard is reproduced on the blackboard.

Systematic Instruction

Following the period of concept-building, systematic instruction in the use and meaning of a few common measures should be undertaken. This systematic teaching is usually begun in

Grade Three. Since children ought to be familiar with the basic ideas of measurement, the first systematic lessons should probably be concerned with demonstrating the need for standard measures. This can be accomplished by having a number of pupils measure the height of some familiar object about the building. In order to prevent the use of standard measures, such as yardsticks or foot rulers, the assignment might be given at a place where no rulers are available — for example, as the children are coming in from some play period. Then at the beginning of the arithmetic period, all would be asked to write a statement about the height of the object that was measured as they came in from the playground.

In one class the height of a tree-limb from the ground was described in these ways: (1) As high as Robert is tall plus my fist. (2) Three and one-half times as long as this stick. (3) As high as my belt is long and then up to the mark on the belt. (4) About six inches higher than I can reach. (5) As tall as Mary plus this stick. On the basis of these reports, special emphasis was given to the necessity for describing the height of the tree-limb so that anyone else, without making use of the same articles or persons that the class employed, could use the statement to find out the exact height. Some of the measures were written on the board and an attempt made to find the best measure. From this point the discussion was properly directed to the need for one measure. Children will usually be able to tell immediately what (foot or yard) this uniform measure should be.

Following a lesson of this type there should be lessons in measuring with standard units. Occasionally unstandardized units, such as the length of the arm, can be used and later translated into standard units Such procedures promote the idea that one unit can be changed to another. They also foster originality and self-confidence in the child.

It should not be necessary to show for every type of measure

the need for uniform measures through use of original or unstandardized ones. However, lessons on the historical development of several measures can make a distinct contribution to the child's understanding and appreciation of modern measures. While these historical lessons might best follow closely the few experiences which are designed to show the need for uniformity in measurement, they are of major importance and may therefore be used at other times. Whether the significance of measures will be really understood from a brief historical description is doubtful. Certainly experience with non-uniform measures will make the historical data more meaningful. Attempts to use ancient measures always make for interest and probably for better understanding of common measures.

The measurement of area is so confusing to children that it demands special attention. If there is any phase of arithmetic that should be introduced through the use of non-uniform measures, it is this one. Children of Grades Five and Six can find the number of square feet in a certain area, but a great many of these same children do not realize that it requires as many squares of the size indicated to cover the surface in question. Even children of Grades Seven and Eight do not have a clear notion of what expressions of area mean. Proof of these statements is found in analysis of children's responses to such questions as the following: "What is meant by saying that a floor contains 650 square feet?" Only 44 per cent of 50,000 sixth-, seventh-, and eighth-grade children chose the correct response, "That it would take 650 blocks, each one foot square, to cover the floor." Seventeen per cent indicated the answer, "That there are 650 squares to the foot of floor space"; while 14 per cent thought the question meant, "That the length, width, and thickness of the floor have been multiplied together."

A good way to give children an idea of what the term *area* means is to have each pupil use duplicate sheets of notebook paper or other material to cover the top of a table or desk. By

counting the smallest number of sheets that will cover the table exactly, the pupil can make a description of the *area*, the size of the space. It is to be expected that different children will use sheets of different sizes as measures. In fact, it is an advantage for them to do so. The use of non-uniform measures will not only be of value in developing the idea of area, but will also create a situation showing the need for standard uniform measures. Suppose, for example, one child reports that his desk top has an area of six arithmetic books and another reports that a table top has an area of eighteen sheets of paper. The teacher introduces comparison by asking whether the table is three times as large as the desk.

Another way of showing need for uniform measures is to have each child measure some surface at home by covering it with a non-uniform measure and then report his finding to the class the next day without giving a description or identification of the measure used. In order to have a common ground for class discussion, the surface measured should be something common to all homes, as the top of a rectangular table. After different measures of area are reported, the teacher should again introduce comparison by asking some child to compare the sizes of two tables.

When measuring with articles such as books and pieces of paper, some of the more advanced children will apply only one piece of paper to two edges. Such a solution is, of course, good, but to emphasize the idea that area means covering the surface, these children should be asked to show that their solution is correct by covering the entire area. After several problems have been solved and the various ways of solving them have come up for discussion, the advanced children should be given an opportunity to present their method. Here, as in the teaching of other phases of arithmetic, the generalization represented by the adult solution should come *after* the children have had experiences leading to the formulation of the generalization.

In other words, the formula "Area equals length times width" should grow out of the experience of covering surfaces with pieces of paper of uniform size. For purposes of proving or demonstrating the truth of solutions, a special set of inch squares and foot squares of manila or cardboard paper should be prepared. Then, after the developmental period, pupils should frequently be required to show by the use of cardboard squares or diagrams that their solutions are correct.

Teaching Tables of Measures

In older teaching methods, the tables of measure were considered essential to the subject of measurement In fact, much of American arithmetic prior to the time of Colburn, and to some extent since, consisted in memorizing various commercial tables and then working problems involving these measures. Modern society, however, requires only infrequent use of such knowledge, and for a number of years less and less emphasis has been given to learning tables of measure. This change in practice seems wise. The few who will need to know tables in their lifework will quickly learn these on the job.

Of course, there are certain facts, such as the number of inches in a foot, that need to be memorized. Since less time will be given to learning tables of measure, the pupils should be taught where to find tables of measures and facts about measures. The dictionary and the arithmetic book are the two main sources to stress. To accustom children to use these sources habitually, many relevant exercises should be employed as part of the program in all the upper grades.

Metric System

Many of the difficulties encountered by the student of arithmetic in his study of measures would be eliminated if the metric system were used. For example, the relationships between various English measures of length, weight, volume, and the

like, are neither consistent (the rod is five and one-half times the yard and the gallon eight times the pint) nor of a ratio that makes for easy computation. On the other hand, the metric measures of length, weight, and volume are either exact decimal fractions of the one major unit or are exact multiples of ten of the major unit. Unfortunately, the metric system has not been accepted very rapidly in America, and therefore too much attention should not be given to teaching it in the schools. Nevertheless, it seems that children would benefit from some familiarity with the meter, centimeter, kilometer, kilogram, and liter.

The reluctance of Americans to adopt the metric system is not entirely a matter of conservatism or of preserving the vested interest. A few metric measures are far more scientific than practical. For example, the kilogram would be a rather awkward weight to use for such staple foods as butter, and the liter would be a poor measure for gasoline. Man finds it difficult to deal with quantities that involve groups of units or groups of collections larger than four or five. To partition a kilogram into the much-used quantity represented by a quarter-pound of butter, an eighth or ninth would have to be used — a division into more units than can be readily grasped. The use of the liter would involve the purchase of twenty units in order to get the equivalent of the common purchase of five gallons of gasoline.

Many of the English measures were adopted because they had already passed the test of practical usage. On the other hand, the metric system was devised on another basis. To the scientist, the centimeter and meter may be useful and sufficient measures; but the man in the street finds the one too small and the other too large to express the lengths of from two to four feet with which he most commonly deals. Measures developed as man experienced a need for expressing things in simpler terms and without relationship to other measures. The most familiar

objects and distances were used as units of measure If the use of a certain unit of measure required a larger number of applications than could be easily grasped, a larger unit of measure was substituted. Thus, for measuring distance we have the inch, foot, yard, rod, furlong, and mile. Regardless of the reason why we keep the English system of weights and measures, it is the system in use, and therefore should be the system receiving major emphasis in schools.

There are some who argue that use of the metric system of weights and measures would result in a better understanding of numbers by pupils and would improve the development of their number ideas. Such a claim for metric measures or any other measures is not based on facts. Measures do not effectively bring out number relationships or number ideas. On the other hand, number ideas and number relationships are indispensable to the efficient use of measures. The major point to remember, as in the case of problem-solving, is that measurement is not used in arithmetic for the purpose of teaching number, but that numbers and number work can be made significant through use of measure.

In spite of all the above considerations, the writer recommends teaching some phases of the metric system as a part of arithmetic, for two reasons: (1) The metric system is widely used in science and in many foreign countries (2) The metric system illustrates the best method for showing relationships between various measures, such as the centimeter as one hundredth of a meter, the millimeter as one thousandth of a meter, and the kilometer as one thousand meters; whereas the similar English units of length are a tenth of an inch, an inch, a yard, and a mile. The metric system uses the meter as the one standard of linear measure; the English system uses several standards — inch, yard, mile — so that the relations between them are rather difficult to grasp. On the other hand, it should be noted that knowledge of the numerical relationships between measures is

not too important for getting an idea of the quantity expressed by each. Consider, for example, whether a knowledge of the length of one foot and of the fact that 5280 feet equal one mile helps in giving any idea of the length of one mile Even if a decimal plan of expressing the relationship between a foot and a mile were available, knowing the length represented by one foot would be of little value in visualizing a mile. A workable concept of the two distances is the result of working with each distance separately. In the metric system, however, it is much easier to comprehend the relation between two linear measures than in the English system. Compare, for example, the meter and the kilometer with the yard and the mile. The ratio of a thousand to one is simpler than the ratic of seventeen hundred and sixty to one. Furthermore, since all metric measures use the decimal relation, an idea of relationships gained with one kind of measures would be applicable to others. The use of one unit throughout the range of measures in the metric system is a perfect illustration of the simplicity that results when organization or system is intelligently used.

Method to Be Used in Teaching Measurement

A brief description has already been given of the method used in teaching measurement of the area of rectangular surfaces. The measurement of areas of other geometrical figures like the triangle and circle should be taught in the same way. A problem involving measurement of the particular surface should be presented, and the children should be told to solve it and then to prove their solution by the use of drawings or by the actual application of measures to the surface. As the final step, the best solution should be selected and written with brief symbols. Later, as problems are worked to insure familiarity with the generalization, frequent proof should be required. The entire procedure is illustrated in the solution of the following problem.

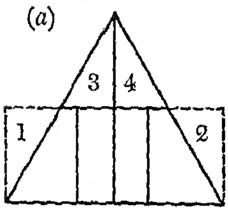
On Mr. Hale's house there are three dormer windows. The space above each window is triangular in shape. He wishes to paint these triangular spaces with a special non-fading paint. In order to know the amount of paint to buy, he must know how many square feet of surface there are in each dormer. If the dormer is 3 feet wide and 2 feet high, how many square feet of space are there in it?

Work this problem in any way you can. After you have a solution, prove that your answer is correct by means of actual measurement of the surface of a drawing of the triangular space, or by means of a drawing. Some of you should use both methods.

After the problem had been solved as directed, each pupil was told to write a brief statement of the solution, and, if possible, to write a rule that could be followed in finding the area of triangles. There was no discussion of the various answers, proofs, methods of solution, or general rule for solution until most of the class had completed all the steps required.

Among the various proofs and methods of solution, the following are representative:

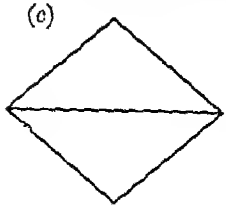
(*a*)



About 3 sq. ft. because the spaces marked 1 and 2 would together equal 3 and 4.

(*b*) Full-size drawing and then the application of foot squares and half-foot square to the surface.

(*c*)



Putting two together, measuring sides, and then figuring as rectangle. Child experienced difficulty because of fractions. Used 3×3, but then subtracted 1 because side wasn't 3. Gave 4 as the answer. (This solution was rejected by the class.)

The rule for finding the area of triangles grew out of the assignment and discussion of the following statement: "In finding the area of rectangles you multiplied two numbers. Since you are using the same square measures to cover the surface here, it seems as though you ought to be able to do the same thing. In fact, you can. See if you can find the two numbers to multiply for every problem we have solved."

As a result of studying these lessons on the proof of measurement problems, children should learn that all measurement is relative and therefore involves error. This fact will be especially evident when finding the relation between the diameter and circumference of circular bodies.

From the preceding sections the reader can easily deduce the major reasons for including measures in the teaching of arithmetic. In the first place, measures are a necessary part of modern life, but it is numbers that make it possible for us to grasp the relations between units of measure and the amounts of whatever those measures are used to describe. Although some amounts to be measured are themselves units and therefore require no use of number to describe them, most amounts are expressed by numbers as multiples of certain units of measure. The field of measurements is one in which numbers have to be used. A second reason for including measurement in arithmetic is that the story of the development of weights and measures is an important part of our heritage. This story cannot adequately be understood unless children have some experience with measures. This second reason for including measurement in arithmetic is relatively untouched in present-day programs. The main argument for including measures then rests with the first reason, which is very important, since almost every important issue of life involves some measurement.

The principles of instruction illustrated in finding the areas of rectangles and triangles apply also in teaching how to find the volume of various geometric figures, circumferences, perimeters,

and the relation between diameters and circumferences. In every case the child should be confronted with a problem situation which he should attempt to solve before the accepted best solution is presented. The best solution should develop from the children's experiences in solving problems.

After the student has accepted the best solution, he should engage in a period of intensive study similar to that followed in learning the basic addition processes. During this period of intensive study the child should be asked occasionally to prove that he understands the process he is using.

Units for Standards of Reference

In the section on the metric system it was pointed out that having a good idea of the distance "one foot" was not of much value in getting an idea of the distance "one mile," even though the relation between the two distances is known. The ratio of 1 : 5280 is not very usable because the average person is unable to see or to think of more than four or five objects without the aid of grouping or counting. Workable concepts of "one foot" and "one mile" are acquired from experience with each length and without much, if any, attention to the relation between the two lengths. In like manner, having a good idea of the distance "one mile" will not be of much value in visualizing or getting the meaning of ninety miles; nor will knowing something of the magnitude of one ton help much in grasping the meaning of ten thousand tons. Children's books (notably geography, history, and science texts), newspapers and magazines, radio programs (especially news programs), and everyday conversation contain many examples of quantities expressed as large multiples of a standard measure. If such expressions are to be understood by children, then the arithmetic program should include more than just the meaning of the standard unit — the foot, pound, and the like. It should provide a set of standard references in

the various fields of measurement that will help children to get meaning from the measurement situations with which they are confronted. For example, in the statement, "The pioneers traveled fifteen miles that day," the distance traveled, although accurately given from the quantitative point of view, would be hard for the child to interpret if the mile were used as the standard of reference. To demonstrate the difficulty, the reader has only to try to visualize fifteen separate miles or to try to visualize the distance by thinking the first mile, then a second mile next to the first, and so on until fifteen miles are reached. A unit larger than one mile must be used as the standard if thinking about the total distance of fifteen miles is to be made easy. As a reference unit for such thinking, a distance of ten miles should be used.

To make these standards of reference function in getting a better idea of the quantity involved, every child should have many experiences and exercises with the special units that are to be used as standards. For example, the distance ten miles should be some clearly defined local distance, such as that to a neighboring town or to some point of interest. If possible, the children should travel this distance by car or bus, should note the time required to ride that distance, figure the time required to walk it, the time required for a fast airplane to fly that far, and the number of times that distance is greater than some short familiar distances. Of course, all these exercises or experiences need not and probably should not come in a relatively short time. Distance concepts, like other concepts, develop slowly.

In addition to the ten-mile distance, another large unit distance of about one hundred miles should be used. The arguments for using the ten-mile unit also apply to the hundred-mile unit. The latter unit should function up to 900 or 1000 miles. Since most distances practically confronting us are seldom greater than that, it is doubtful whether the provision of another reference unit would be worth while. Then, for linear distances

other than the foot, yard, and one hundred yards, three references or standards for comparison are needed. These are the mile, ten miles, and one hundred miles. The real-life standards need not be exactly ten or one hundred miles. In fact, if a large city or other important point were 90, 125, or 150 miles away, it would be better to use one of those distances as a unit of reference.

For every phase of measurement, similar standard reference points should be developed. Thus, for area, the acre, quarter-section, square mile, and area of one's state would be used. The children should be provided with experiences similar to those described for the ten-mile standard, in order that they may gain a better idea of the quantity represented by each standard reference Of course, the procedures used to give an idea of the size of a state would vary because of the nature of the measure. In this case the time required to travel across it, the distance around it, comparison of some of its dimensions with standard reference distances, and other similar procedures would add to the children's conception of the size of their state.

The value of an area reference like a state can be seen by considering the following extract from a news item: "The region involves an area of 85,000 square miles." The person who knows the area of his own state is about 50,000 square miles can quickly deduce that the region under consideration is approximately one and one-half times as large as his own state. On the other hand, the person who has no such standard of reference either fails to get a usable idea of the area or else takes time for reflection which interferes with his understanding of the other ideas in the news item.

The study of standard references might well be a part of such content subjects as history, geography, or health, but since the number part of each measurement is common to all, it seems best to include standard references in arithmetic rather than to have a separate number-concept section as a part of each sub-

ject area. Then, too, statements involving distances, areas, altitudes, and so on, have been a part of geography for many years, and to date no simple set of references for the child to use has been developed through the teaching of geography. It is through the provision of such references that arithmetic can really become an aid to thinking.

Since most arithmetic programs contain little in the way of a systematic plan for teaching quantitative reference points, special attention should be given to methods of teaching this phase of arithmetic. Beginning with Grade One, a few standard references should be developed each year. While most of the units of reference will be collections of measures, such as 100 miles, 10,000 tons, and 50,000 square miles, the teaching program should probably begin with the basic unit — that is, the mile, the ton, and the square mile. For these basic units a reference object or distance is used. As was stated before, children should have many experiences with each reference and frequently should be asked to make comparisons using the reference. For example, the pound is one of the references used in some first grades. The first need or discussion of the pound usually arises as a result of a visit to the store, the market, or the creamery. On their return to the classroom, the children usually make statements about their experiences, and these statements almost always involve the word *pound*. Some actual objects, like a pound of butter or loaf of bread weighing a pound, may be brought back; but, for the most part, familiar classroom objects will be weighed. As the teacher writes on the board a statement about the price of a pound of meat, she might well ask if anyone can find something in the room that weighs about a pound. The children then select objects, weigh them, and mark one as the reference pound. Many times during the year they will refer to this pound and make comparisons with it.

In the upper grades the need for references is often demonstrated through an oral problem, such as the following: "I read

that the poinsettias in a Florida garden were from 12 to 15 feet high. How high do you think that is?" The reference suggested for use in such a case is the height of the classroom door. Through oral exercises, not only are many of the standard references demonstrated, but also facility in the use of references is maintained.

Much research is needed to determine the best list of standard references for inclusion in the arithmetic program. The following list is taken from the tentative course of study in arithmetic at the University of Iowa Elementary School: [1]

Grade Two

(*a*) Seven feet — the height of the classroom door.
(*b*) Thirty feet — the length of the classroom
(*c*) Ten pounds — the weight of small oak chair.

Grade Three

(*a*) All references from the preceding grade.
(*b*) An acre — a plot near the school grounds, of which the dimensions are measured and memorized (about 200 feet on a side).
(*c*) A mile — the distance from the school to Iowa City Poultry Company Plant.
(*d*) A bushel — a bushel apple basket.

Grade Four

(*a*) All references from the preceding grades.
(*b*) A square mile — the area bounded by the following streets: Davenport, Capitol, Benton, and Grant-Parsons.
(*c*) Ten miles — the distance from Iowa City to West Branch.
(*d*) 120 miles — the distance from Iowa City to Des Moines.

Grade Five

(*a*) All references from the preceding grades.
(*b*) 30 inches — the height of doorknob, teacher's desk, or dining table.

[1] *Arithmetic in the University School* (Iowa City, Iowa. State University of Iowa, 1941).

(c) 50 feet vertical — the height of the southwest corner of Currier Hall.
(d) 3 tons — amount of coal in local coal truck; 35 tons — amount of coal held by standard railroad coal car.
(e) 50,000 square miles — the area of Iowa. (For practical reasons the exact area of 56,147 square miles is rounded to 50,000 square miles.)
(f) A city of 20,000 people — the population of Iowa City.

Grade Six

(a) All references from the preceding grades.
(b) A city of 150,000 people — Des Moines.
(c) 9000 miles of railroad — the number of miles of railroad in Iowa.
(d) 14,000 feet elevation — the elevation of Pikes Peak.
(e) 10,000 tons — the capacity of a large freighter, or iron ore boat. (The equivalent of approximately six freight trains of fifty cars each.)

Teaching How to Find Averages

In dealing with different statements of measures of similar quantities, such as the ages, heights, or weights of children in elementary classrooms, the pupil can think more efficiently if he selects one typical or representative measure than if he tries to think of all the measures. The mean or average, the mode, and the median are the three types of representative measures that are used. Of these, the average seems to be the only one with which arithmetic is concerned. This emphasis on one type of measure probably stems from the fact that arithmetical programs have not been concerned with the meaning of such a representative measure, but rather only with the numerical operations that are necessary to compute it. Since the average is the only one of the three that takes into consideration the exact quantity of every measure in the group of measures, it would probably have been selected for emphasis in arithmetic even if meaning had been considered. Regardless of the

reason for the extreme emphasis on averages, modern arithmetical programs do in fact devote a great deal of time to the finding of averages. Some attention should therefore be given to the matter of how best to teach this phase of arithmetic.

In most series of texts, the first problems on finding the average are placed in the fourth grade. Following the usual plan in introducing new phases of arithmetic, a problem is used to show where averages are found and how they are found. The text problems deal with areas of life so universal in nature and the explanations appear to be so clear-cut that one can hardly see how pupils can miss the meaning of averages. The usual textbook method of teaching, however, does not give the child the experience that will enable him to see *why* averages are used. Furthermore, even though the explanations seem unusually clear, they fail to convey to the pupil the idea (at least in concrete form) that averages are obtained by an "evening-up" process; that is, the short measures are built up while the longer ones are cut down. (See lesson below.)

The method recommended for teaching averages and directing the child's first learning experience in this phase of arithmetic is illustrated by the following:

> Mary and Harry had an argument about who had the tallest rose bushes. To settle the argument they measured the bushes to the nearest foot. They found that Harry had the tallest one, but that he also had the shortest. Mary claimed that, on the average, hers were taller. Here are the heights of Mary's four bushes: 7 feet, 6 feet, 6 feet, and 6 feet. Harry's three bushes measured 8 feet, 6 feet, and 4 feet. Was Mary right? Show how you thought, so that you can prove to the others that you have the correct answer.

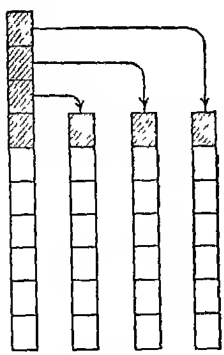
No further directions were given to the class as a whole. To those who did not seem to be able to start, the teacher suggested that they draw a diagram to represent the height of each bush. She even suggested to one child that the heights be

marked on the blackboard. Since some children did not know the meaning of the term *average*, the class work was interrupted in order that this idea might be considered. The chief explanations of the children for the word were: (1) "It's when you put them all together and divide by the number you added." (2) "It means make all the same just as if all were the same height." (3) "Like in spelling you add all scores and then divide by as many as are in class." (4) "It's not the biggest or littlest but in between." The teacher said that these statements were all true; but because she felt some of them might not be too clear, she added: "When you average quantities you try to make them all the same by taking some of the larger quantities and adding to the smaller ones."

When the children went back to work, the teacher wrote on the board another problem involving average. In this way additional time was gained for the slower children to work on the first problem.

In the discussion period that followed, most of the children agreed that Mary was correct. Several methods of proof were offered, all of them variations of these two plans.

(*a*)



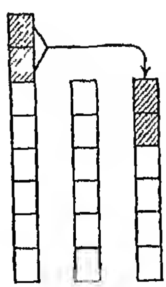
Mary's

(*b*) $7 + 6 + 6 + 6 = 25$

$25 \div 4 = 6\frac{1}{4}$

$8 + 6 + 4 = 18$

$18 \div 3 = 6$



Harry's

In method *a* the extra length of the first bush was divided into four parts. Three of the parts were put on the six-foot bushes

while one was left on the first. Harry's bushes were equalized in the same way. Method *b* is self-explanatory. The major points considered in discussing the solution were: (1) does it agree with the facts of the problem? and (2) does it show who has on the average the tallest bushes? In order to emphasize the fact that the average was a representative measure, one standing for the others, the teacher several times asked this question about a solution: "How can you say that Mary's bushes averaged six and one-fourth feet when she didn't have any that were exactly that tall?"

This problem involved an unequal number of measures in the two groups that were being compared, a condition chosen in order to avoid the child's use of total height as a means of comparison. The comparison situation was used because it illustrates better the reason for using averages. Ordinarily, problems involving averages are concerned with finding the average only. A few problems of this type may be used during the introductory phase of the teaching of averages. In considering such problems, the average as a representative measure should be emphasized.

Several days after the first problem with averages had been dealt with, the following problem was given·

Four boys were having a football kicking contest. Each boy was allowed five kicks. The boy whose record showed the longest average kick was to be the winner. John kicked these distances: 15 yards, 17 yards, 8 yards, 16 yards, and 14 yards Jack had these: 12 yards, 10 yards, 15 yards, 15 yards, and 16 yards. Henry had these: 20 yards, 16 yards, 5 yards, 15 yards, and 9 yards. Tom had these: 14 yards, 14 yards, 16 yards, 13 yards, and 12 yards. Who was the winner?

In addition to answering the question raised in the problem, the children were asked to "prove that your answer is correct and tell why it is better to consider the average kick than it is to try to compare the individual kicks." In the discussion that

followed completion of the assignment, the children gave this reason for considering the average kick: "It is easier to think of one number than it is to try to think of all the numbers."

In the teaching plan described above, the chief purpose of instruction was to give children an opportunity to see what averages are and the reason for using them. After such an initial period of instruction, some time should be taken for intensive work on problems involving the finding of averages. This type of work helps to fix the process of finding averages.

Since the finding of averages is so common in social studies and in science work, it is hardly necessary to consider a maintenance program in this area of arithmetic. However, teachers in the upper grades should continually check the children's understanding of the term and of the reason for its use.

STUDY QUESTIONS

1. How many different kinds (volume, length, and so on) of measures comprise the measurement in arithmetic? *(1)* 4. *(2)* 5. *(3)* 6. *(4)* 7.

2. Are a good idea of the distance "one foot" and a knowledge of the fact that 5280 feet equal one mile of any value in visualizing how long a distance a mile is? *(1)* Yes. *(2)* No.

3. Measures are said to be substitutes for the real thing. Is that true of measures of weight (e.g., a pound determined by use of a spring balance)? *(1)* Yes. *(2)* No.

4. For which of these different measures is it best to use the standard measure during the concept-building period? *(1)* Length *(2)* Volume. *(3)* Weight. *(4)* N.

5. What is the purpose of using unstandardized measures in the pupils' initial experience with an area of measurement (for example, linear or surface)? *(1)* To demonstrate a need for standardized measures. *(2)* To demonstrate the fact that measurement can be done without standard measures. *(3)* To show the relationship between measures. *(4)* N.

6. In teaching measurement of area which of these ideas is most important? (*1*) That length times width gives area. (*2*) That area is a matter of covering the entire space. (*3*) That measures of area are squares. (*4*) N.

7. What is the best argument for not attempting to have pupils master tables of measures? (*1*) The pupil has no use for this knowledge in school (*2*) The trend in arithmetic is away from memorization. (*3*) Tables are usually not understood; hence the learning is little short of verbalism. (*4*) N.

8. The statement is sometimes made that knowledge of the relationships between various measures given in a table, as, for example, in a table of length, is of little value in getting a good idea of a specific measure. Is that true? (*1*) Yes. (*2*) No.

9 Americans have not accepted the metric system readily because they already have a well-developed system. Is there any other good reason? (*1*) Yes. (*2*) No.

10. What is the major shortcoming of the metric system of weights? (*1*) The base weight is too large (*2*) The base weight is not a practical size. (*3*) Similarity of names (*4*) N.

11. Is it possible to prove that the solution of a measurement problem is correct by actual measurement? (*1*) Yes. (*2*) No.

12. What is the intended function of the standard reference measures in this book? (*1*) To provide a foundation for the measures to be taught in arithmetic. (*2*) To provide an aid for interpreting measures. (*3*) To provide an aid for interpreting tables of measures. (*4*) N.

13. What idea or fact about averages is the most important to an understanding of averages? (*1*) That averages are exact, based on every measure used. (*2*) That the average is always less than the largest measure and more than the smallest. (*3*) That the average is frequently about the middle one. (*4*) N.

14. Why is it better to use the average than to use the numbers from which the average has been computed? (*1*) The average is more accurate. (*2*) It is easier to use the average in thinking than it is to use the numbers (*3*) Most people understand averages better than they do the individual numbers.

9

Common Fractions

Illustrative Teaching

The development of meaning and understanding is as important in teaching the addition of fractions as it is in teaching the addition of whole numbers. In keeping with the recommended plan of this book, the procedures described below employ (1) problems which the children can solve even though inefficient methods may be used; (2) various means of solving the problems in order to arrive at the best method, (3) proof to demonstrate understanding of the best method; and (4) a period of intensive study.

The children of a fourth-grade class had reached the stage where they were ready for the systematic study of fractions. To begin this work, the following assignment was given: "In your map project you are using many fractions. In checking your measurements I find that I frequently add. You know how to handle fractions, but I have noticed that you have several ways of doing it. In arithmetic today, let's work these fraction problems to find out the different ways of solving that the class uses. You may learn a better way of handling fractions or perhaps you will show someone else a better way. Work these problems, using diagrams, and then, if you can, show with numbers what you did with the diagrams."

1. Harry was drawing a map on which he had marked the points A, B, C, and D in this way:

The line from A to B was $2\frac{1}{4}$ inches long; the line from B to C, 1 inch long; and the line from C to D, $2\frac{1}{4}$ inches long. How long was the line from A to D?

2. The four berry plants in Mr. Bane's back yard produced the following amounts of berries last year: 2 pints, $2\frac{1}{2}$ pints, 3 pints, and $1\frac{1}{2}$ pints. How many pints of berries were produced by the four plants?

3. One of the rats in pen No. 2 of our health project weighs $1\frac{1}{8}$ pounds and the other weighs $1\frac{3}{8}$ pounds. In order to be able to feed the rats the proper amount of food, the class must know the total weight of the rats. How much do the two together weigh?

As the children solved the problems, the different methods in use were noted by the teacher. To children who used only numbers in their solutions, the teacher recalled the original suggestion that they make a drawing or diagram. However, since these children already had answers to the problem questions, the drawings were to be means of proving that the solutions were correct. To others the teacher suggested the use of the actual measures in problem 1. In order that the slow workers might have time to get experience with as many problem situations as possible, the teacher told some of the rapid workers that they might try to find other methods of solving the problems, and suggested that the circle is often used in solving fractions. Later some of these special solutions were put on the board. The drawings that occurred most frequently in the various solutions were linear representation, rectangular representation, and the circle. For problem 2, realistic drawing of pints was also used. When the different solutions were evaluated, correctness of presentation and clearness were the only

criteria applied. In this evaluation the relation between diagrams and number solutions was stressed. It was pointed out also that the number solutions were good if you understood them, since they told in a short way what was done with the diagrams.

For the next set of problems the following assignment was given: "First, work the problem by using a diagram or actual measures. Then, tell with numbers what you did in solving the problem." As the children worked with the new problems, the teacher suggested to individual children who had already used one type of diagram — for example, linear representation — that they use another type, such as the circle or the rectangle. After a work period, the number solutions were evaluated and the best method selected from all those used. Each proposed method was tested by seeing if it gave the right answer when used in solving several problems that had already been worked out by the use of drawings. After the method had passed this test, the following rule for adding fractions was developed. "When you add fractions, you add only the numerators and use your old denominator for the sum."

The teacher pointed to the next step by stating that a method had now been found, but that such methods were usually soon forgotten unless something was done to fix them firmly in mind. The class then discussed ways of fixing the method so that it would not be quickly forgotten. A number of things, such as "Work more problems," "Take tests," and "Study hard on several problems," were suggested. The teacher agreed that all these plans were good, but thought that the working of addition exercises like those in the textbook should certainly be one of the ways of fixing the method. She pointed out that the children could use the other proposals, but suggested that some time during the next few days might well be spent in working the examples in the textbook. During this period of intensive study, occasional examples were worked by means of

the circle technique or one of the diagram methods. Proof by these means was given special emphasis through such statements as, "You know you can always work it with diagrams," "That is the real test of the correctness of your work," and "That is the only way I can tell whether you understand what you are doing."

Purpose and Characteristics of Fractions

The procedure just described illustrates some methods which emphasize understanding. The steps followed by this one class, in its first lessons on the addition of fractions, cannot, of course, serve as the only guide to the teaching method. The brief description, however, did raise some of the issues met in the teaching of fractions. Other issues are raised and discussed in this section.

Like so many other phases of arithmetic, fractions will be much more easily grasped if their uses are understood. Fractions have these four major functions: (1) indication of one or more of the equal parts of a whole or a unit; (2) indication of one or more of the equal parts of a collection; (3) indication of an unperformed division; (4) expression of a ratio. The first two of these uses of fractions are a part of the concept-building program recommended for Grades One and Two. For an example of the type of exercise that will develop these two meanings of fractions, see exercise 20, Chapter 4. The third use of fractions does not appear until division with remainders is encountered, and the fourth probably does not arise until the sixth or seventh grade is reached.

Fractions are almost as old as whole numbers themselves. From the meager records that have come down to us, it is clear that fractions have been very difficult to master. Part of this difficulty has resulted from the fact that in handling a fraction two concepts have to be dealt with simultaneously. Whereas

in comprehending a whole number one needs to consider only one number, in comprehending a fraction attention must be given to two separate numbers, one of which indicates the size of the fractional part and the other the number of parts. This difficulty is well illustrated by reviewing briefly some attempts of the ancients at simplification of fractions.

To enable him to give attention both to the number of parts and to the size of the part, as in the fraction $\frac{3}{8}$, ancient man devised a scheme whereby such a fraction would be transformed into unit fractions. Three-eighths then became $\frac{1}{4}$ and $\frac{1}{8}$. The calculator thus needed to focus his attention only on the size of the part, since all numerators were one. Perhaps such a procedure is a simplification step in the case of $\frac{3}{8}$; but when a fraction like $\frac{4}{5}$ has to be expressed as four $\frac{1}{5}$'s, or as $\frac{1}{2}$, $\frac{1}{5}$, and $\frac{1}{10}$, a grasp of the fractional quantity is undoubtedly made much more difficult.

Another scheme used by the ancients in an effort to overcome the difficulty of fractions involved making all denominators alike, so that attention need be fixed only on the numerator. Babylonian sexagesimal fractions (sixtieths) are a good example of such a procedure By this method all fractions were expressed as sixtieths. Any remainder left over, after the fraction was expressed in sixtieths, was expressed as a second sixtieth. Thus, $\frac{3}{8}$ became 22 first sixtieths and 30 second sixtieths. That this scheme actually worked in eliminating the necessity of thinking about denominators is shown by our own system of minutes and seconds, measures of time that correspond to the first and second sixtieth parts of the Babylonians. The terms "minute" and "second" come from the Roman parts *minutia* and *second minutia*. In this sexagesimal scheme we may also see the germ of our system of decimal fractions. Our decimal fractions, however, do not drop the size concept from the expression of the fractions. A close study of both unit and sexagesimal fractions will show that each method of expression, while aiming for

simplicity, may actually complicate the situation. In order to get the full meaning of fractions, both size and number of parts should be considered.

The denominator of a fraction gives the size of the part under consideration. Of course, this size of the part may be determined by dividing a whole into as many parts as is indicated by the denominator. Few children, however, see any relation between the common definition of denominator (the number of parts into which the whole is divided) and the size of the part. Much confusion results from trying to teach children that the denominator tells the number of parts into which the whole is divided. Consider, for example, the fraction $\frac{1}{8}$. There are not eight parts but only one part, which has the size of an eighth. If the fraction were $\frac{3}{8}$, there would be three, not eight, parts, each of which has the size of an eighth. It is the numerator, not the denominator, that gives the number of parts. That many pupils of Grades Six, Seven, and Eight fail to comprehend the meaning of the numerator is shown by their responses to this test question. "What does the numerator of a fraction tell you?" Four possible answers were listed:

(1) How many fractional parts are taken.
(2) The size of the unit to which the denominator refers.
(3) What to use in reducing the fraction.
(4) Whether to add, subtract, multiply, or divide.

The percentages on the responses chosen by the pupils were as follows: (1) 48%, (2) 35%, (3) 10%, and (4) 4%. Three per cent of the children tested failed to answer.

Fractions in the First Four Grades

Systematic instruction such as that described at the beginning of this chapter should be preceded, of course, by work with fractions in the concept-building program and with fractions that arise in the everyday school work of the earlier grades.

Directions such as "Use about one-third of your page for the drawing," should be given frequently.

The concept-building program should be concerned with the presentation of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and other fractions that occur incidentally. The first work should be with parts of actual objects, or of measures like cup, quart, or pound. Later these fractions should be used in connection with such measurements as inches, feet, and miles. In Grades One and Two the work should be done orally and demonstrated with concrete materials. If any record needs to be made, the word form (one-half) rather than the number form ($\frac{1}{2}$) should be employed. However, since measuring cups are marked in the familiar number-form manner of writing fractions, many first- and second-grade teachers prefer to use that form. In Grade Three or Four, wherever division is introduced, the child will have use for the number form of writing fractions, and that form should then be employed. In order to insure presentation of the proper meaning of this numerical way of writing fractions, the introductory problems used should deal with things that are logically divisible, such as apples, pies, and cookies. In the solution of such problems, the pupil can see the reason for the division indicated by the fraction. Later, as children write quotients with fractional parts, they should sometimes be asked to show with drawings or objects what the fractions signify. The fact that a fraction as used in such cases is an indication of division should be continually emphasized.

As a part of the experience program in arithmetic in the primary grades, children will sometimes be required to add and subtract fractional amounts in solving a problem. Situations of this type need not be avoided. The fractions can be solved by use of the ruler, the measuring cup, or whatever measure is required by the problem. Through the use of actual measures, it is possible to avoid the difficulties which adults encounter with addition and subtraction of fractions that have unlike

denominators. The work with fractions in the first four grades is concerned primarily with developing meaning. The teaching of adult computational procedures is assigned to later grades.

It should be noted that textbooks give some space to the development of the fraction concept. In practically all instructional plans, the teaching of computation with fractions is preceded by a concept-building program. Almost every series of textbooks introduces fractions in Grade Three, and further attention is given to them in Grade Four before addition of fractions is started. These textbooks provide excellent procedures for teaching the meaning of halves, thirds, fourths, and sometimes a few other fractions. The experiences include work with units, such as half an apple, and with collections, such as half of six. While the textbook exercises are excellent for developing the fraction concept, they need to be supplemented by first-hand experiences such as the following: (1) finding by actual measurement that a pint is one-half of a quart; (2) showing by measurement that in a half-filled vessel the empty space is the same size as the filled space; (3) finding through change-making that one dollar is equal to four quarter dollars; (4) finding by comparison that each of the fourths of an apple is equal in size to any of the others.

Initial Instruction: Addition of Fractions

As was the case in the procedure described at the beginning of this chapter, the first systematic instruction directed toward the short method of computation with fractions should be concerned only with addition and should describe a situation in which measuring of quantities can be performed. Since the first use we have enumerated for fractions (the indication of one or more parts of a unit) is easier than the second (the indication of one or more equal parts of a collection), the introductory

problems should deal only with situations involving this first use. Children can be asked, therefore, to prove their solutions by using measures, such as the ruler and the cup, rather than collections of objects, such as apples, blocks, and the like.

In the procedure described, initial work consisted of the addition of mixed numbers ($3\frac{1}{4} + 2\frac{1}{4}$) in which the fractions had like denominators. In common practice initial instruction deals with the addition of a fraction to a fraction ($\frac{1}{4} + \frac{1}{4}$). Since the fractions to be added in these two instances are the same, the method using mixed numbers certainly appears to be the more difficult of the two. Then, why use mixed numbers in initial instruction? There are two reasons for doing so. First, the most common type of problem in the addition of fractions, and therefore the one which will be most familiar to children, involves mixed numbers. Second, it is generally held that the familiar whole-number part of the mixed-number addition situation gives the child more confidence in his ability to get the sum. The child knows that six and six equal twelve, and therefore he can see that six and one-fourth plus six and one-fourth are a little more than twelve.

When children start systematic work in the addition of fractions, it should be remembered, they are already very familiar with the basic idea of addition with whole numbers, and they have already had much experience with fractions. The use of mixed numbers for initial instruction is, then, not a case of presenting to the child a totally new situation. Furthermore, it should be noted that the mixed numbers are used for the introduction to the process and other types of fraction addition are used in the study of the process. Types of examples in the addition of fractions need not be presented in any fixed order. since all types will be used by the child in the course of a few weeks' instruction. The following list of examples illustrates adequately the five types used in the addition of fractions.

ADDITION OF FRACTIONS: ORDER OF PRESENTATION

A. Whole Numbers and Mixed Numbers or Fractions

$1\frac{1}{4}$	$3\frac{1}{2}$	4	6	$2\frac{3}{4}$	$5\frac{7}{8}$	3	4
$+\ 2$	$+\ 1$	$+\ 2\frac{1}{3}$	$+\ 1\frac{1}{8}$	$+\ 2$	$+\ 1$	$+\ 2\frac{2}{3}$	$+\ 7\frac{3}{8}$

$\frac{1}{4}$	2	3	$\frac{1}{2}$	2	$\frac{2}{8}$	$\frac{1}{3}$	$\frac{7}{8}$
$+\ 2$	$+\ \frac{1}{3}$	$+\ \frac{3}{4}$	$+\ 4$	$+\ \frac{2}{3}$	$+\ 5$	$+\ 4$	$+\ 4$

B. Similar Fractions (Sum one or less)

$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{5}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{2}{5}$
$+\ \frac{1}{4}$	$+\ \frac{1}{4}$	$+\ \frac{1}{3}$	$+\ \frac{3}{8}$	$+\ \frac{1}{8}$	$+\ \frac{1}{2}$	$+\ \frac{1}{8}$	$+\ \frac{1}{5}$

C. Similar Fractions (Sum greater than one)

$\frac{3}{4}$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{7}{8}$	$\frac{5}{6}$	$\frac{7}{8}$	$\frac{3}{5}$
$+\ \frac{3}{4}$	$+\ \frac{2}{5}$	$+\ \frac{4}{5}$	$+\ \frac{2}{3}$	$+\ \frac{3}{8}$	$+\ \frac{5}{6}$	$+\ \frac{7}{8}$	$+\ \frac{3}{5}$

D. Fractions with Unlike Denominators

$\frac{1}{2}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{1}{3}$	$\frac{3}{4}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{4}$
$+\ \frac{1}{4}$	$+\ \frac{3}{4}$	$+\ \frac{1}{2}$	$+\ \frac{1}{8}$	$+\ \frac{2}{5}$	$+\ \frac{3}{4}$	$+\ \frac{7}{8}$	$+\ \frac{2}{3}$

E. Mixed Numbers

$2\frac{1}{4}$	$15\frac{1}{2}$	$5\frac{3}{4}$	$8\frac{2}{3}$	$6\frac{1}{4}$	$3\frac{3}{4}$	$8\frac{1}{3}$	$5\frac{4}{5}$
$+\ 3\frac{1}{2}$	$+\ 10\frac{1}{2}$	$+\ 16\frac{7}{8}$	$+\ 5\frac{2}{3}$	$+\ 2\frac{2}{3}$	$+\ 7\frac{3}{4}$	$+\ 9\frac{2}{3}$	$+\ 6\frac{2}{5}$

Type A, addition of a whole number and a fraction or of a mixed number and a whole number, is so elementary that few textbooks so far simplify the presentation to include that type in the problems and examples for study. Type B, addition of fractions of like denominators whose sums are one or less than one, is the type most frequently used in beginning instruction. Type C is often combined with type B. In type C the sums are greater than one. In type D unlike denominators occur.

Type E has already been pointed out as the commonest situation involving addition of fractions. In the program described at the beginning of this chapter, this type E was used in initial instruction. Since the only phase of the addition of mixed numbers that is new to the child is the addition of the fractions, the major part of a child's work should be given to types B, C, and D.

In the addition of fractions with unlike denominators, problems new to the pupil arise. One has to do with recognizing that denominators have to be the same before addition is possible. The other has to do with a method of finding a common denominator.

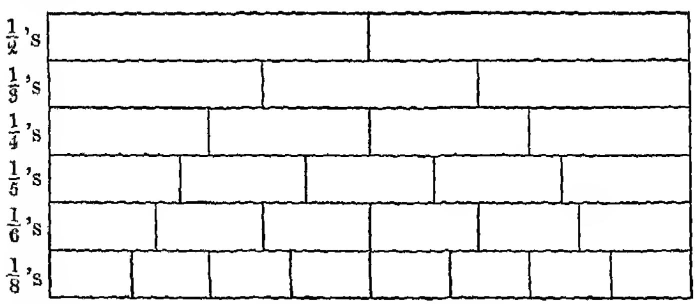
In preparation for work with common denominators, most instructional programs include work on equivalence of fractions. Rulers, measuring cups, charts, diagrams, folding and cutting of rectangular pieces of paper and the like are used in showing such equivalent fractions as $\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$. To demonstrate that unlike fractions cannot be added in the usual pencil-and-paper way, a problem involving either addition or subtraction is often used. As has been advocated for the introduction of other new phases of arithmetic it is suggested that the pupil be permitted to find a solution before the best solution is supplied by text or teacher. In the first addition and subtraction situations one of the denominators is the common denominator, and therefore the pupil has only to change one fraction. For example, when $\frac{1}{2}$ is to be added to $\frac{1}{8}$, the pupil should think "one-half is equal to how many eighths?" The following way of writing such examples is considered good:

$$\begin{array}{rcl} \frac{1}{2} & = & \frac{4}{8} \\ +\ \frac{1}{8} & = & \underline{\frac{1}{8}} \\ & & \frac{5}{8} \end{array}$$

When fractions with unlike denominators of the type $\frac{3}{4} + \frac{2}{3}$ (common denominator not visible) are encountered, the finding

of a common denominator has to be done in a manner different from that just described. Two ways of finding the common denominator are frequently employed. The first may be called the *method of inspection*. By this method the pupil tries to select after inspection of the denominators given a denominator common to both. The second procedure for finding a common denominator in such situations as $\frac{3}{4} + \frac{2}{3}$ consists in multiplying one denominator by the other. While this *product of denominators* method often results in use of a common denominator other than the lowest common denominator, the method has merit, and for the pupil not too proficient in arithmetic it is usually better than the inspection method. Many pupils who are taught the product of denominators method soon use a combination of that method and the inspection method.

In teaching either method of finding the common denominator, the instructor should provide situations where a need for the procedure is shown. A chart of the type shown below will be



useful when this new phase of fraction work is introduced. The ruler may also be used to advantage in giving a background for finding a common denominator. If chart or ruler is used to add or subtract unlike fractions, the problem "Is there a number method of making these denominators the same?" might be an excellent way to lead up to the presentation of the best way of finding common denominators.

Subtraction of Fractions

Teaching of the subtraction of fractions, as has already been indicated, should follow rather than accompany the teaching of addition of fractions. The same principles used in teaching the addition of fractions are applicable to the teaching of subtraction of fractions, and for that reason no description of teaching procedures will be presented here.

Most textbooks begin the presentation of fraction subtraction with the subtraction of a fraction from a fraction. Here, as in addition, the use of mixed numbers for initial teaching has some merit. In fact, two leading textbooks do use mixed numbers along with fractions in initial presentation. The types of situations used in the teaching of subtraction of fractions are as follows.

SUBTRACTION OF FRACTIONS: ORDER OF PRESENTATION

A. Whole numbers subtracted from mixed numbers

$$\begin{array}{r} 8\frac{1}{4} \\ -\ 2 \\ \hline \end{array} \qquad \begin{array}{r} 3\frac{3}{4} \\ -\ 1 \\ \hline \end{array} \qquad \begin{array}{r} 8\frac{2}{3} \\ -\ 4 \\ \hline \end{array} \qquad \begin{array}{r} 6\frac{1}{3} \\ -\ 3 \\ \hline \end{array}$$

B. Similar fractions

$$\begin{array}{r} \frac{2}{3} \\ -\ \frac{1}{3} \\ \hline \end{array} \qquad \begin{array}{r} \frac{3}{4} \\ -\ \frac{1}{4} \\ \hline \end{array} \qquad \begin{array}{r} \frac{5}{8} \\ -\ \frac{1}{8} \\ \hline \end{array} \qquad \begin{array}{r} \frac{9}{10} \\ -\ \frac{3}{10} \\ \hline \end{array}$$

C. Fractions with unlike denominators

$$\begin{array}{r} \frac{3}{4} \\ -\ \frac{1}{2} \\ \hline \end{array} \qquad \begin{array}{r} \frac{7}{8} \\ -\ \frac{1}{4} \\ \hline \end{array} \qquad \begin{array}{r} \frac{2}{3} \\ -\ \frac{1}{2} \\ \hline \end{array} \qquad \begin{array}{r} \frac{5}{6} \\ -\ \frac{1}{4} \\ \hline \end{array}$$

D. Mixed numbers without borrowing

$$\begin{array}{r} 14\frac{1}{2} \\ -\ 6\frac{1}{4} \\ \hline \end{array} \qquad \begin{array}{r} 8\frac{2}{3} \\ -\ 5\frac{1}{2} \\ \hline \end{array} \qquad \begin{array}{r} 6\frac{3}{4} \\ -\ 2\frac{1}{3} \\ \hline \end{array} \qquad \begin{array}{r} 7\frac{7}{8} \\ -\ 1\frac{3}{4} \\ \hline \end{array}$$

E. Mixed numbers with borrowing

$$\begin{array}{r} 17 \\ -\ 8\frac{1}{2} \\ \hline \end{array} \qquad \begin{array}{r} 14\frac{1}{2} \\ -\ 10\frac{1}{2} \\ \hline \end{array} \qquad \begin{array}{r} 12\frac{2}{3} \\ -\ 4\frac{7}{8} \\ \hline \end{array} \qquad \begin{array}{r} 10\frac{3}{4} \\ -\ 4\frac{5}{6} \\ \hline \end{array}$$

Since the process of finding a common denominator has already been encountered by the pupil in his study of the addition of fractions, the only essentially new thing to be learned in the subtraction of fractions is the borrowing or changing of a whole number to a fraction. Even this process is not basically new since a related process is used in the subtraction of whole numbers. The actual procedure to be used in showing borrowing and the change to a common denominator is an important factor to many teachers. Most people when subtracting $4\frac{7}{8}$ from $12\frac{2}{3}$ write the example as follows:

$$\begin{array}{r} 12\frac{2}{3} = \frac{40}{24} \\ -\ 4\frac{7}{8} = \frac{21}{24} \\ \hline 7\frac{19}{24} \end{array}$$

Of course, such people know that $12\frac{2}{3}$ does not equal $\frac{40}{24}$. To avoid such misrepresentations the whole number should be rewritten; that is, $12\frac{2}{3} = 11\frac{40}{24}$.

Although the additive method of subtraction is not recommended by the author, some attention should be given by the student of the teaching of arithmetic to its use in the subtraction of fractions. In the example

$$\begin{array}{r} 7\frac{1}{8} \\ -\ 4\frac{5}{8} \\ \hline \end{array}$$

the thinking suggested by some of those who use the additive method is as follows: 5 and what number of eighths will equal eight eighths, or one? The answer is 3. This 3 eighths and the 1 eighth equal $\frac{4}{8}$, the required remainder. 5 and 2 equal 7. The answer is $2\frac{4}{8}$ or $2\frac{1}{2}$. It appears to the author that the rationalization of this additive procedure would be more difficult for the child than would be the rationalization of the take-away method of subtracting fractions.

Multiplication of Fractions

In the multiplication of fractions these five main types of examples are used:

A. Fractions multiplied by integers $(3 \times \frac{1}{2})$
B. Integers multiplied by fractions $(\frac{1}{3} \times 4)$
C. Mixed numbers multiplied by whole numbers $(4 \times 2\frac{1}{2})$
D. Fractions multiplied by fractions $(\frac{1}{2} \times \frac{3}{4})$
E. Mixed numbers multiplied by mixed numbers $(3\frac{1}{2} \times 2\frac{1}{4})$

It should be recognized that it is possible to classify multiplication examples into many other types; in fact, some writers use as many as forty types. A study of the examples of the five types (see Order of Presentation, below) will show how such an extensive classification is possible Almost every one of the thirty examples presents some new difficulty. For instance, under type A we have in example 1 a unit fraction $(\frac{1}{2})$; in example 2, a fraction larger than a unit fraction $(\frac{2}{3})$; in example 3, a product that is either an improper fraction or a mixed number; in example 4, a fraction larger than a unit fraction and a mixed number for the product; in example 5, the product is the multiplicand; and in example 6, the product is a fraction.

MULTIPLICATION OF FRACTIONS: ORDER OF PRESENTATION

A.	$\frac{1}{2}$ $\times 4$	$\frac{2}{3}$ $\times 6$	$\frac{1}{3}$ $\times 5$	$\frac{3}{4}$ $\times 3$	$\frac{1}{3}$ $\times 1$	$\frac{3}{8}$ $\times 2$
B.	4 $\times \frac{1}{2}$	3 $\times \frac{1}{2}$	3 $\times \frac{2}{3}$	4 $\times \frac{1}{8}$	1 $\times \frac{1}{4}$	2 $\times \frac{3}{8}$
C.	$2\frac{1}{2}$ $\times 4$	$3\frac{3}{4}$ $\times 4$	$3\frac{2}{3}$ $\times 5$	4 $\times 2\frac{1}{2}$	4 $\times 2\frac{3}{4}$	4 $\times 6\frac{2}{3}$
D.	$\frac{1}{4}$ $\times \frac{1}{3}$	$\frac{1}{4}$ $\times \frac{1}{4}$	$\frac{2}{3}$ $\times \frac{1}{2}$	$\frac{1}{4}$ $\times \frac{3}{8}$	$\frac{3}{4}$ $\times \frac{2}{3}$	$\frac{3}{8}$ $\times \frac{3}{8}$
E.	$4\frac{1}{2}$ $\times 2\frac{1}{2}$	$3\frac{1}{2}$ $\times 2\frac{1}{4}$	$5\frac{3}{4}$ $\times 2\frac{2}{3}$	$3\frac{3}{8}$ $\times 4\frac{1}{8}$	$6\frac{1}{2}$ $\times 5\frac{7}{8}$	$7\frac{3}{5}$ $\times 1\frac{3}{4}$

It should be noted that type C can be subdivided into two distinct types, $4 \times 2\frac{1}{2}$ and $2\frac{1}{2} \times 4$. These two are treated as one type because most people interchange results in a simplified procedure. Since this procedure of interchanging is recommended, we shall consider the two types as in fact one.

The order of presentation, as given here, is not universally accepted. Many texts use type B for the introductory systematic work in multiplication. The finding of unit fractional parts of numbers, such as one-third of twelve, is taught as a part of the division of whole numbers in most textbooks. Those that use the same situations in teaching multiplication attempt to relate the new procedure of multiplication of fractions to the division of whole numbers. In order that children may see a need for the multiplication procedure, problems are used that involve the finding of other than unit fractional parts; for example, finding two-thirds of twelve. But even with the need for multiplication made evident by the use of non-unit fractions, the multiplication of a fraction by an integer seems the better type to use for the introductory work. It is not so easy to find two-thirds of twelve as to find twelve two-thirds. To illustrate, consider the two following problems:

1. Each day John used two-thirds of a pound of feed for his pigeons. How much feed did he use in twelve days?

2. Two-thirds of the twelve bushels of apples were graded number 1. How many bushels of number 1 apples were there?

Other differences of opinion on the order of presentation will be found if arithmetic texts for children are examined. However, since none of the differences seems crucial, no further discussion is offered here. Indeed, all the types have to be taught in a relatively short time.

MULTIPLICATION OF FRACTIONS: TEACHING PROCEDURE

Beginning instruction in the multiplication of fractions gives to problems a very important role. As is true in the introductory teaching of other processes in arithmetic, the first problems used ought to be of such a type that they can be solved by the use of drawings and other indirect means. From these indirect solutions the number method of multiplying fractions is worked out. The illustrative teaching procedure which follows gives the main steps.

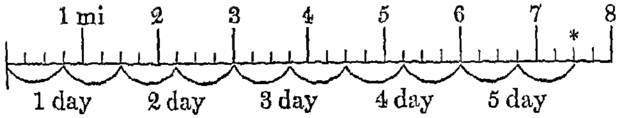
The directions for the first assignment in the multiplication of fractions were. "Show with a drawing or a diagram how you can get the answer to the question in each problem. Use numbers to label only. Be sure to tell the whole story of the problem in your diagram and try to make it clear to anyone who may look at it."

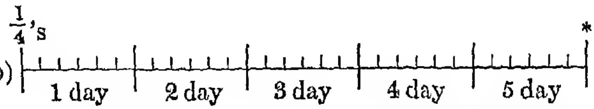
1 It is three-fourths of a mile from Mary's home to school. In five days how far does she walk going to and from school?

2. Jack used a half-gallon bucket to fill the water keg. He poured sixteen bucketfuls into the keg. How many gallons did Jack pour into the keg?

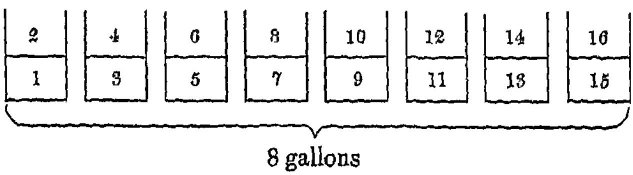
As the children worked on these and four or five similar problems, the teacher went about the room occasionally asking a question concerning some illustration and giving hints to those who were unable to start or who had made a mistake. These two are typical of the hints given. (1) "Why not draw a line about an inch long for a mile and then mark it off in fourths?" (2) "Where do you show the amount poured into the keg the first time Jack emptied the bucket?"

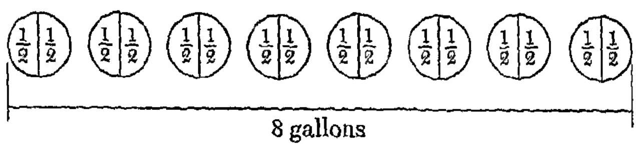
During the latter part of the work period several of the pupils were asked to put their diagrams on the board. Representative diagrams for problems 1 and 2 are reproduced here.

1. (a) 

(b) 

$* = \frac{30}{4} = 7\frac{1}{2}$ miles

2. (a) 

(b) 

As soon as one pupil had finished the diagrams for all the problems, an evaluation of the diagrams and answers for the first two problems was started. In this evaluation, the diagrams of the different members of the class were discussed. The criteria used in judging a diagram and the answer secured were: (1) Does it illustrate the problem? (2) Is it easy to see (to read)? (3) Does the answer agree with the facts shown in the drawing?

After the evaluation period, pupils resumed work on the problems. Those who had finished were asked to make dia-

grams that could be used in answering more questions of this type:

1. What is the weight of seven packages if each weighs three-fourths of a pound?

2. Six one-thirds are equal to how many?

The second period devoted to multiplication of fractions was concerned with finding a number way of solving the problems that had already been solved by means of diagrams. This assignment was given the class. "For each of the problems in your last lesson, write a number solution. Use your diagrams or the diagrams on the board." As the pupils worked, these additional directions were given by the teacher: "When you have a number solution, try to decide whether it is the best number solution for that problem. A good way to decide is by trying to write another solution for the same problem."

The three most common number solutions offered by the pupils for problem 1 were:

(1) $10 \times \frac{3}{4} = 7\frac{1}{2}$

(2) $$\begin{array}{r} \frac{3}{4} \\ \times\ 10 \\ \hline 7\frac{1}{2} \end{array}$$

(3) $\frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} + \frac{3}{4} = \frac{30}{4} = 7\frac{1}{2}$

The three most common number solutions for problem 2 were:

(1) $16 \times \frac{1}{2} = 8$

(2) $16 \div 2 = 8$

(3) $$\begin{array}{r} \frac{1}{2} \\ \times\ 16 \\ \hline 8 \end{array}$$

It should be recalled that the answer and a diagram for each problem were available to the pupils, so that their only task was the formulation of a number statement of the solution. Therefore, all answers were correct and attention was directed

toward choosing the best number solution. In making this choice two things were considered: (1) What do you do with the numbers to get the answer? (2) Is that the shortest number solution? The first question is concerned with understanding of the process; the second, with economy of time. A glance at the first number solution for each problem will show that knowledge of the answer and of the numbers used in getting that answer is not enough to demonstrate to the pupil how to get the answer with numbers. The second number solution for problem 2 was explained by a pupil in this manner: "It's half of sixteen and you get half of a number by dividing by 2." The pupil was using what he had learned in partition measurement. At first glance this learning appears to be a handicap in developing the process of multiplying fractions. The situation may be avoided by using other than unit fractions in these introductory problems. To help the pupils to see how $7\frac{1}{2}$ is the answer secured by multiplying $\frac{3}{4}$ by 10 (see number solution 1, problem 1), the teacher suggested that the $\frac{30}{4}$ as shown in solution 3 be inserted as follows: $10 \times \frac{3}{4} = \frac{30}{4} = 7\frac{1}{2}$. "Now, what do you do with the 10 and $\frac{3}{4}$ to get $\frac{30}{4}$?" was the next statement used to direct the thinking of the class. As soon as a satisfactory procedure had been accepted, the same plan was used with the other number solutions, and then the rule for multiplying a fraction by a whole number was worked out. The next assignment for this class consisted of a few problems to be solved by the use of both number solutions and diagrams, followed by a list of practice examples. As the pupils worked the examples they were occasionally asked to show by means of diagrams that their work was correct.

The teaching plans used in developing the four other types of multiplication examples are similar to that just described for the multiplication of a fraction by an integer. In every case problems are used to present the new situation, the solution for the problem is obtained by use of diagrams or other indirect

solutions, the best number method is worked out, a rule is formulated, and then practice is used to fix the process. If pupils know how to use the number procedure, they are permitted to do so, but are then asked to show by diagram or other methods that their number solutions are correct.

The rule developed for the multiplication of fractions of type A examples ($4 \times \frac{3}{8}$) is applicable to type B examples ($\frac{2}{3} \times 6$), and can be used with little modification for type C ($4 \times 3\frac{1}{2}$), but it is inadequate for types D ($\frac{1}{2} \times \frac{2}{3}$) and E ($3\frac{1}{4} \times 6\frac{1}{5}$). In those multiplication situations in which only one fraction is involved, the denominator in the product is the same as the fraction multiplied. When two fractions are multiplied the two denominators are multiplied, a procedure very different from any which the child has thus far encountered in his work with fractions. The required new procedure is easily related to that already worked out for the multiplication of a fraction by an integer. The rule is then changed from "In multiplying a fraction by a whole number you multiply the numerator by the whole number and use the denominator" to "In multiplying fractions you multiply one numerator by the other numerator and one denominator by the other denominator." To show that this last rule will cover all examples of the multiplication of fractions, pupils should be required to use it in multiplying such examples as $4 \times \frac{2}{3}$ and $\frac{3}{4} \times 5$. "What is the other denominator?" or a similar question is usually sufficient direction to enable pupils to see the need for writing the whole number as an improper fraction with 1 for a denominator.

The multiplication of a mixed number by a mixed number requires that pupils learn and understand an efficient way to change mixed numbers to improper fractions. The usual plan of developing the best method from diagrams or other indirect methods is recommended for teaching the procedure for changing mixed numbers to improper fractions. The best method is, of course, to multiply the whole number by the denominator

of the fraction, add to the product the numerator of the fraction, write this sum as the numerator of the improper fraction, and use the original denominator for the denominator of the improper fraction. The best method is illustrated in the solution below.

Example: Change $7\frac{3}{8}$ to an improper fraction.

Step 1. $8 \times 7 = 56$. (Multiply the whole number by the denominator of the fraction.)

Step 2. $56 + 3 = 59$. (To the product found in step 1, add the numerator of the fraction.)

Step 3. $\frac{59}{8}$. (Use the sum from step 2 for the numerator of the improper fraction, and for the denominator use the original denominator.)

Division of Fractions

Examples in the division of fractions are easily classified into three main types:

A. Whole numbers divided by fractions
B. Fractions divided by whole numbers
C. Fractions divided by fractions

The use of mixed numbers would add a subdivision to each of these three main types. Since mixed numbers will have been used in the teaching of multiplication of fractions, their introduction here would not involve anything especially new to the pupils. For that reason, only the three types of the division of fractions listed above are considered as distinctly separate.

DIVISION OF FRACTIONS: ORDER OF PRESENTATION

On the order of presentation current practice as represented by textbooks is about evenly divided between type A and type B for first position, with all texts placing type C last. In view of this divided opinion, special study of types A and B should aid the student in making a choice. Type A, involving the division of a whole number by a fraction, can be introduced by use of a measurement problem. Type B, involving the division

of a fraction by a whole number, can be introduced satisfactorily only by means of a partition problem. The following problem is a good example: "Four boys found that there was only a third of a pie to be divided among them. If each received an equal share, what was the size of the piece of pie which each boy received?" In such a situation the division by actual cutting of the pie or by a representative drawing does not make for as easy recognition of the answer as does actual manipulation or representative drawing in the case of the measurement problem. It should be recalled that, in the division of whole numbers, measurement division was considered less difficult than partition division. With fractions the difference in difficulty of the two types of division is even more marked. In the partition situation with fractions, a single unit or thing, not a collection of units as is true of whole numbers, is divided. The illustrative problem cited above shows how difficult such division can be. The type B problems used in textbooks in introducing division usually avoid the difficulty just mentioned by using only situations in which the numerator of the fraction to be divided and the whole number divisor are the same. Not infrequently, teachers ask why the division of a fraction by a whole number (type B) cannot be introduced by means of a measurement problem. The answer to the question is made obvious by asking the essential measurement question, such as "How many fours in two-thirds?"

From the facts presented in the preceding paragraph, it seems that in a program which places emphasis on understanding, type A, the division of a whole number by a fraction, is the one to use in initial instruction. Type A certainly lends itself better than type B to the kind of developmental procedure advocated by the writer. Type A is also superior for programs which introduce division of fractions by the common denominator method. (See Teaching Procedure for description of this method.)

DIVISION OF FRACTIONS: TEACHING PROCEDURE

The procedure to be used in teaching the division of fractions is similar to that used in teaching the multiplication of fractions. Briefly, the steps are as follows: problems involving the division of whole numbers by fractions are provided; the solutions are obtained by means of diagrams or other indirect methods; number solutions are worked from these indirect solutions; the proposed number solutions are tested, and by means of practice examples the accepted best methods are fixed. Since the most difficult point in the division of fractions is the changing of the situation from division to multiplication by inversion of the divisor, the teaching of this part of the procedure should receive special attention.

In the procedure outlined above, inversion of the divisor occurs when the class works out a number solution. As indicated in the discussion of the multiplication procedure, the answer to a problem in the division of fractions will have been secured by means of diagrams or other indirect methods before an attempt is made to work out a number solution. The thinking of the pupils is directed by such comments as the following: "From your diagrams you have found that 6 divided by $\frac{2}{3}$ equals 9. What can you do with the numbers 6 and $\frac{2}{3}$ to get the answer 9?" No logical addition or subtraction of the whole number and the fraction will result in the 9. The two solutions that will give the desired answer are: (1) divide 6 by 2 and multiply the resulting quotient by 3; and (2) multiply 6 by 3 and divide the resulting product by 2. In each solution there is a multiplication, but the multiplication involves the numerator of one term and the denominator of the other. If the pupils do not point out the conflict between such a procedure and the procedure worked out for the multiplication of fractions, the teacher should do so. "How can we change the fraction so that we multiply the numerator by the whole number as we do in the multiplication of fractions?" then becomes the problem of the

class. "Turn it over" and "Turn it upside down" are two suggestions that pupils are likely to offer. After the procedure has been tested the teacher usually suggests using the one word *invert* in place of the three words *turn it over.*

The development of the procedure for inverting the terms of the divisor in the preceding paragraph was approached through analysis of a number statement of a diagram solution. The learner's thinking was directed by the problem: "What can I do with these numbers to get this answer?" Another way of developing the inversion procedure is shown below through the solution of problems involving the division of whole numbers and fractions.

Illustrative problem 1 If 3 boxes of pepper are each of the same weight and all together they weigh 18 ounces, what does each box weigh?

Solution: The weight of each box is equal to the total weight divided by 3, or $\frac{1}{3}$ of the total. The total weight is 18. Then each weighs $\frac{1}{3}$ of 18 ounces. $\frac{1}{3}$ of $18 = \frac{1}{3} \times 18 = \frac{18}{3} = 6$. The weight of each box is 6 ounces.

Illustrative problem 2 If $\frac{3}{4}$ of a box of oatmeal weighs 2 pounds, what will a full box weigh?

Solution: The weight of a full box (each box) is equal to the weight of $\frac{3}{4}$ of a box (the total weight given) divided by $\frac{3}{4}$, or to $\frac{4}{3}$ of the weight of $\frac{3}{4}$ of a box. The total weight, or weight of $\frac{3}{4}$ of a box, is 2 pounds. Then a full box will weigh $\frac{4}{3} \times 2 = \frac{8}{3} = 2\frac{2}{3}$ pounds.

In the solution for the first problem, $18 \div 3$ was shown to be the equivalent of $\frac{1}{3} \times 18$. Therefore, $18 \div 3 = \frac{1}{3} \times 18$, or $18 \times \frac{1}{3}$. In the solution for the second problem $2 \div \frac{3}{4}$ was shown to be the equivalent of $\frac{4}{3} \times 2$. Therefore, $2 \div \frac{3}{4} = 2 \times \frac{4}{3}$. The procedure may be expressed in words as follows: To divide by a fraction, invert the terms of the divisor (the fraction) and multiply.

The development of the inversion procedure from the solution of problems may have value in clarifying the procedure for

those who are already using it, but it is difficult for pupils who are just beginning the process. If children are to see sense in studying the two solutions which are to show that in dividing you invert the terms, then an assignment similar to the following will be needed: "Why do you invert the terms of the divisor and multiply when you divide by a fraction?"

Consideration of the two ways of developing the inversion procedure used in division by a fraction should point out the difficulty of rationalizing the procedure. The fact that inversion is difficult to understand is one argument for placing emphasis on proof and for having pupils try to figure out the inversion procedure. The division of fractions by the accepted adult method has been so difficult for elementary-school pupils that some authorities propose substitutes for the best method or complete omission of the process One of the substitute methods of dividing fractions is described in the next paragraphs.

Within the last fifteen years the common denominator method of introducing division of fractions has received a great deal of attention. The method is illustrated in the teaching procedure which follows:

Illustrative problem: How many sticks one-half foot in length can be cut from a 5-foot stick?

Solution: You can get the answer to the question by using a ruler and finding how many one-half foot lengths there are in five feet When you do that, you show that $5 \div \frac{1}{2} = 10$

Another way to find how many halves in five is to change the five to halves and then divide. Here is the way you do it:

$$5 \div \tfrac{1}{2} = \tfrac{10}{2} \div \tfrac{1}{2} = 10 \div 1 = 10$$

To show you how to divide in this way notice how 8 is divided by $\frac{3}{4}$.

$$8 \div \tfrac{3}{4} = \tfrac{32}{4} \div \tfrac{3}{4} = 32 \div 3 = 10\tfrac{2}{3}$$

This changing to like fractions (common denominator) and then dividing numerators only is used with all the other types of examples of division of fractions. The common denominator

method has some very commendable features. Division of fractions by this method is division of whole numbers and with that students certainly should be familiar. The changing to a common denominator has good precedent in addition and should, therefore, appear as a reasonable procedure to the pupils. In fact, when division examples are solved by diagram the dividend and divisor are actually changed to like fractions. For example, when you find by diagram how many eighths in three-fourths, you change the three-fourths to eighths and then count the eighths. The common denominator method has stood up remarkably well in small experimental studies, but it has not been tested on a wide scale.

In texts that use the common denominator method, the transition from that method to the inversion method of dividing fractions is made rather abruptly, with very little done to bring out the relations between the two methods. The two methods, however, can be easily related, and inversion developed directly from the common denominator method. To illustrate, consider the example $\frac{3}{8} \div \frac{2}{3}$. This would be solved as follows by the common denominator method:

$$\frac{3}{8} \div \frac{2}{3} = \frac{9}{24} \div \frac{16}{24} = 9 \div 16 = \frac{9}{16}.$$

The thinking of the pupils can then be directed by some such statement as the following: "You have shown that $\frac{3}{8} \div \frac{2}{3} = \frac{9}{16}$. Can you see a short way of getting $\frac{9}{16}$? Try to find one." The situation is then very similar to the one used in developing the number solution from the diagram solutions. For a description of that procedure see page 278.

What Fractions to Teach

A great deal of research has been devoted to determining the fractions most used in life. This research has shown that most adult usage is confined to $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$,[1] with $\frac{1}{8}$, $\frac{1}{12}$, and $\frac{1}{16}$ occur-

[1] G. M. Wilson and others, *Teaching the New Arithmetic* (New York, McGraw-Hill Book Company, 1939), pp 194–212.

ring next in frequency. Several writers have suggested that only $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, because they comprise the bulk of fraction usage, should be taught in elementary-school arithmetic. If that suggestion were followed, the fractions program would be rather meager and probably much more difficult for the child to learn than is the case where $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$, $\frac{1}{12}$, and $\frac{1}{16}$ are included. Practice exercises confined only to $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ would certainly have to be brief or contain much repetition. For teaching purposes the writer suggests that $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{8}$, and $\frac{1}{10}$ be used; and if other, less common fractions appear to be beneficial in establishing a principle of addition, subtraction, or multiplication, the teacher should not hesitate to use them. The important thing to keep in mind is that the program of teaching should aim to develop the *principles* of adding, subtracting, multiplying, and dividing fractions. It is not enough merely to teach addition, subtraction, multiplication, and division of halves, thirds, and fourths. The pupil who has learned the principle can add the common fractions with more confidence than can he who has learned to add only the few common fractions, without much emphasis on principles. If the pupil is to understand a process, the nature of that process becomes important. As a means of illustrating the nature of processes, fifths are probably as useful as fourths. It might be well to note the fact that the number of times a fraction occurs in adult usage is of little concern to a fifth-grade child who is learning to add fractions.

Research findings have shown not only that adult usage is confined primarily to a few fractions, but also that the process of subtraction with fractions is not used very much and the process of division of fractions is used infrequently. On the basis of these findings it has been suggested that the teaching of subtraction be very limited and that the division of fractions be omitted. In the opinion of the writer, it would be a mistake to follow these suggestions. If fractions are to be understood thoroughly, each of the four fundamental processes has to be

mastered. Furthermore, many occasions arise that call for division of a whole number by a mixed number. The best way to teach pupils how to do such division is to develop the whole process of dividing fractions. An additional reason for teaching division of fractions is that all processes are needed in higher mathematics — an argument that has been severely criticized on the ground that uses in future courses should be postponed until those courses are taken But when it is remembered that a similar argument could be advanced for much of the arithmetic taught in the lower grades, such reasoning seems less valid.

Since fraction problems do not occur as frequently as problems with whole numbers, the maintenance and re-teaching program should be carefully planned. In order to provide opportunities for understanding, proof should be used occasionally as a part of the maintenance exercises; that is, pupils should be asked to show with diagrams how $\frac{3}{4}$ and $\frac{2}{3}$ are added; how $\frac{3}{4}$ of $\frac{2}{3} = \frac{1}{2}$; how $\frac{3}{5}$ are subtracted from $1\frac{1}{2}$, how $\frac{2}{3} \div \frac{1}{8} = 5\frac{1}{3}$, and the like. Both re-teaching and maintenance exercises will probably be more effective if examples and problems are used, rather than problems alone.

Three Kinds of Problems with Fractions

To help children solve certain kinds of fraction problems some textbooks have adopted a practice very similar to the "case method" in percentage. By this method fraction problems are classified, and, as in percentage, three kinds or classes of problems are used. The first kind requires the finding of a part of a number; the second, finding what part one number is of another; and the third, finding the whole or total when only a part is given. The three kinds of problems are illustrated in these samples:

Problem 1. One-fourth of the pupils in Grade One were ill with influenza. If there are 28 children in Grade One, how many were ill?

Problem 2. Four of the twelve children were under weight. What part of the children were under weight?

Problem 3. If two-thirds of a load of gravel weighs 5000 pounds, what will a full load weigh?

By the "case method" pupils are taught first to identify the kind or case of a fraction problem and then to use the solution recommended for that kind of problem. For example, to solve problem 1 above (Case I), multiply the number by the fraction. For problem 2 (Case II) divide the number representing the part by the number representing the whole. For problem 3 (Case III), divide the part given by the fraction.

The value of classifying fraction problems and then using the recommended procedure for each kind has not been determined by experimental studies or by long usage. The case method will probably never be as popular with fractions as with percentage, for the situations are basically different. Whereas practically all percentage problems can be classified according to these categories, not nearly all fraction problems are of the three kinds. In fact, most fraction problems involve only addition or subtraction.

Some teachers question the value of the case method of solving fraction problems because they think it tends toward a mechanical procedure with little understanding of the processes used. The opponents of the case method contend that the pupil, instead of considering the total problem situation, first looks for cues which will enable him to identify the case-type, and then, as soon as the case is determined, looks only for the numbers to use in the recommended solution. The process of identifying problems is questioned also because it seems to introduce an extra step between the careful reading of the problem and the formulation of a possible solution. The teachers who oppose the use of the case method with fraction problems are not agreed as to what procedure should be used. By far the majority of teachers recognize no special plan for

solving fraction problems, but treat such problems in much the same manner as whole-number problems; that is, a general method of solving is used. Since there are types of whole-number problems which parallel closely the three kinds of fraction problems, the solution of fraction problems by the same general method appears to have good precedent. An examination of the parallel whole-number problems will aid in clarifying the issues in this general procedure.

Problem 1. "How many pencils are there in 7 boxes if each contains 6 pencils?" Here the essential question to be asked is "7×6 is what number?" — a question very similar to "$\frac{2}{3} \times 12$ is what number?" — the essential question of a Case I fraction problem

Problem 2 "Pumpkin A weighs 12 pounds and pumpkin B weighs 4 pounds. How many times as heavy as pumpkin B is pumpkin A?" Here the essential question is "12 is how many times 3?" This question is similar to such Case II fraction problem questions as "4 is what fraction of 12?"

Problem 3. "How many pieces of candy can Nancy give to each of 4 girls if each is to receive the same amount and she has only 12 pieces?" Here the essential question to be answered is "4 times what number is 12?" This question is very similar to Case III questions like "$\frac{3}{4}$ of what number is 12?"

The important issue regarding the case method in whole-number problems is well summed up as follows. In answering the essential question for each of the three problems above, is a special solution dependent upon identification of the kind of problem necessary or beneficial? The question asked in the first kind of problem (Case I) indicates the operation to be performed, and therefore identification of the problem in order to determine the operation to be used would be superfluous. The question for the second kind of problem (Case II) is practically the same statement as that used in division examples, and the question for the third kind of problem (Case III) also indicates rather clearly that division is the operation needed for the

solution. Thus, it appears that if the nature of the essential question is determined, the operation to be used is rather clearly implied, and the identification of kinds of problems will not serve any useful purpose. Study will show that the operation to be performed is indicated almost as clearly by the essential questions for fractional problems as for whole-number problems. Consequently, the case method with its dependence upon identification of cases may introduce an unnecessary step in the solution of fraction problems. Before concluding that the formulation of the essential question is superior to the case method, the difficulty of formulating the essential question should be compared with the difficulty of identifying the kind of problem. There seems to be little difference here, since identification is dependent upon determining what you are to find, a step quite *similar to formulating the essential question.*

A few teachers who oppose the use of the case method with fraction problems recommend the use of the equation method for solving such problems. In the equation method the essential question to be answered is first formulated and then the letter n substituted for the unknown and the equal sign used for the word "is" or "are" in the question. For example, the Case I question, "$\frac{2}{3} \times 12$ is what number?" becomes "$\frac{2}{3} \times 12 = n$"; the Case II question, "4 is what part of 12?" becomes "$4 = n \times 12$"; and the Case III question, "$\frac{1}{4}$ of what number is 20?" becomes "$\frac{1}{4} \times n = 20$." While the writing of equations from statements such as the above is a simple procedure, it should be recognized that sixth-grade pupils are not familiar with ways of solving equations. Then, before the equation method of solving can be successfully used, pupils must first learn how to solve such equations. How difficult this would be has not been determined, although some practical suggestions dealing with this problem have been made.[1]

[1] H. Van Engen, "Unifying Ideas in Arithmetic," *Elementary School Journal*, 42: 291–96 (December, 1942).

STUDY QUESTIONS

1. The most common use of fractions is to indicate parts of quantities, either wholes or collections. Which should be presented first? (*1*) Wholes; e g., $\frac{1}{4}$ of 1. (*2*) Collections; e.g., $\frac{1}{4}$ of 8.

2. Why are fractions more difficult to deal with than are whole numbers? (*1*) Because whole numbers are all of the same size. (*2*) In dealing with fractions both size and number have to be dealt with (*3*) Whole numbers are arranged into a system, while fractions are not. (*4*) N.

3. If measures are used in the introduction to addition of common fractions, is it important to have only fractions with like denominators? (*1*) Yes (*2*) No.

4 Are the arguments for the simultaneous presentation of addition and subtraction of basic facts applicable to the addition and subtraction of common fractions? (*1*) Yes. (*2*) No.

5. Which of these types of fractions should be presented first in teaching addition? (*1*) $\frac{1}{4} + \frac{1}{4}$. (*2*) $2\frac{1}{2} + 4$. (*3*) $\frac{1}{3} + \frac{3}{4}$. (*4*) $5\frac{3}{4} + 1\frac{7}{8}$.

6. In teaching multiplication of fractions why is $3 \times \frac{1}{2}$ recommended for initial presentation rather than $\frac{1}{2} \times 3$? (*1*) The first is likely to occur more frequently (*2*) The first is easier. (*3*) The second is entirely a case of partition division. (*4*) N.

7. In the introductory lessons on the multiplication of fractions a diagram solution rather than a number solution was required. Why? (*1*) Because pupils at this stage are really not capable of understanding number solutions. (*2*) Because time for the slow learners is provided by this procedure. (*3*) Because in this way all pupils are put on a more equal footing. (*4*) N.

8. In situations where a fraction has to be divided by a whole number, which type of division (measurement or partition) occurs most frequently? (*1*) Measurement. (*2*) Partition. (*3*) Occurrence is about equally divided.

9. In showing that $\frac{1}{4} \times \frac{1}{6} = \frac{1}{24}$ what type of diagram is best? (*1*) Circle. (*2*) Bar or line. (*3*) Rectangle.

10. What feature of the common denominator method of dividing fractions makes this method easy for children to master? (*1*) The computation is the same as in addition of fractions (*2*) The computation is the same as in division of whole numbers. (*3*) Since both denominators are the same, you divide by only one. (*4*) N.

11. Such fractions as $\frac{1}{5}$, $\frac{1}{6}$, and so on, occur only infrequently in life outside the school. How, then, can their use in teaching children be justified? (*1*) For teaching a principle an uncommon fraction is practically as useful as is a common one. (*2*) They are needed to provide additional practice. (*3*) In the few instances where such fractions occur their use is crucial. (*4*) N.

12. What significant limitation is there in the application of the "case method" to fraction problems? (*1*) A large number of fraction problems do not come under the three cases (*2*) The cases with fractions are not so easy to identify as with percentage. (*3*) The solution requires knowledge of equations, something the children have not yet studied. (*4*) N.

10

Decimals and Percentage

Introduction to Decimal Fractions

Systematic teaching of decimal fractions is usually delayed until after the basic idea of fractions has been developed through the use of common fractions. Because computation with decimals, especially addition and subtraction, is so much easier than with common fractions, many teachers have suggested that decimals be taught first. There are these two good reasons for beginning with common fractions: (1) the basic idea of fractions can be developed more easily with halves, thirds, and fourths than with tenths, hundredths, and thousandths; (2) common fractions provide almost an ideal foundation for the teaching of decimals. In fact, if common fractions and whole numbers are understood well, notation is the only part of addition and subtraction of decimals with which the pupil is not already acquainted.

Long before systematic teaching of decimals is undertaken, most children have some experiences with fractions of this kind. The most common use of decimals is in our money system. Since the money system uses a special name of "cents" for hundredths, it is doubtful if many children realize the fractional aspect of money. The decimals which children most commonly encounter arise out of the measurement of distance and time. Bicycle and automobile speedometers register fractional dis-

tances in tenths of miles; and track, racing, and swimming times are measured in seconds and tenths of seconds. Many data in the fields of agriculture, economics, and health also use tenths and hundredths written as decimals. Thus, in previous arithmetic situations and in life outside of school, there will have been many experiences which should provide a good foundation for the teaching of decimals.

Before turning to teaching procedures some attention should be given to decimal notation, especially with regard to its relation to the notational scheme we use for whole numbers. By writing fractions as decimals we merely apply to the parts of the unit the same plan we applied to the collections of the unit. The tenths correspond to the tens, the hundredths to hundreds, and so on. In a number such as 33.33 each figure represents a quantity ten times as large as that to its right. When explaining decimal notation many teachers and a few textbooks place a great deal of emphasis on the decimal point as a marker or separating line between whole numbers and decimals. Such emphasis is unfortunate because it diverts attention from the real focal point in our notational scheme. The one or the unit is a pivotal, not a separating, point. Figures in all positions to the left of the ones place represent multiples of one, and all figures to the right of the ones place represent fractional parts of ones. If the unit or ones place is emphasized when explaining decimal notation, such questions as "Why isn't there a decimal fraction to correspond to the ones in whole numbers?" will occur less frequently.

Reading and Writing of Decimals

The reading of decimals is usually begun in Grade Five. A situation in which tenths are written as decimals introduces this new work. "John rode his bicycle 1.3 miles" is typical. If no member of the class can read the decimal, the teacher reads it or rewrites the statement, using a common fraction.

Emphasis is placed on the fact that the decimal affords another way of writing three-tenths. A number of exercises are then provided which require the writing of tenths as decimals and as common fractions. In order that pupils may have an opportunity to see the value of learning to read and write decimals, some data from lifelike situations should be written in both the decimal and common fraction form. Good examples of such situations are the recording of yields per acre in agricultural contests or trials, the listing of distances from school to familiar places in the neighborhood, and the listing of the time per lap of great track stars. A glance at two columns, one using decimals and the other using common fractions, will show neatness of appearance and ease in reading as two points in favor of the decimals. Furthermore, pupils who have had the experience of writing the same data both ways will see that decimals save time and effort in writing.

The plan for teaching the reading and writing of hundredths and thousandths as decimals is similar to that used for tenths. It should be noted, however, that as decimals of different denominators are introduced a new difficulty is presented. In common fractions both numerator and denominator are shown as separate numbers. In decimals the same numbers indicate both numerator and denominator. Pupils can be helped to determine the size (the denominator) of a decimal fraction in two ways: by use of common fraction equivalents, and by use of a chart labeling the names of the various positions. These two ways are illustrated below (*a* and *b*):

(*a*) $\frac{1}{10}$ is written	.1	.1 is read one tenth
$\frac{1}{100}$ is written	.01	.01 is read one hundredth
$\frac{14}{1000}$ is written	014	.014 is read fourteen thousandths

There are as many places in the decimal fraction as there are zeros in the denominator of the equivalent common fraction. You can use this to figure out how to read a decimal. First,

write tenths, hundredths, and so on, as common and then as decimal fractions. Keep writing larger denominators until you get one with as many places as the decimal you want to read.

	hundreds	tens	ones	tenths	hundredths	thousandths
(*b*)	3	2	6	5	2	

The number 326.52 is read three hundred twenty-six and fifty-two hundredths

A chart like this can be used to read decimals. Write the figures of the decimal to be read in the proper places on the chart. Then use the name of the place for the last figure for the name of the decimal. In the example the last figure, 2, is in the hundredths place. Therefore, the decimal is read fifty-two hundredths.

The fact that decimals require one less figure than the correspondingly named whole number is often a source of confusion. For example, hundredths require only two figures while hundreds require three. The fact that decimals are parts and cease to exist (become a whole) when the three figures in the case of hundredths are reached is, of course, the explanation to use when pupils are confused by the apparent inconsistency in writing.

Another way which is similar to the chart involves learning the names of the places; that is, tenths, hundredths, and so on.

Addition and Subtraction of Decimals

Addition of decimals precedes the subtraction of decimals in most textbooks, but in practically all programs the two processes are placed in close proximity. A few texts teach the two processes simultaneously. The usual procedure is very similar

to that used with whole numbers. The process is introduced by means of a problem, the solution of that problem by adding decimals is explained in detail, and then examples for learning and fixing the process are provided.

The procedure recommended in this book differs from the above primarily in the introduction. Instead of one problem, several are provided; pupils are to solve the problems either by use of common fractions or diagrams; the answer is changed to a decimal fraction; and then the steps in getting the answer by adding decimals only are figured out. The steps in the procedure are illustrated below.

Assignment: Use common fractions or diagrams to get the answer to the question in each of these problems. When you have the answer, try to figure out how you can get the answer by using decimals only.

Problem 1. Mary lives one and four-tenths miles from school How far does she walk in going to and from school?

Problem 2 Helen's bicycle speedometer read 3.2 miles when she left home. When she got to Jane's house the speedometer read 3 7 miles. How far is it from Helen's house to Jane's house?

After the pupils had found the answer to these and similar problems the explanation of decimal solutions was considered. The pupils' thinking was directed at this stage by such statements as the following: "Write the decimals to be added (or subtracted). Next, change your common-fraction answer to a decimal and write it underneath the decimal numbers you were to add. Now, let's figure out how you can add the decimals to get this answer." In learning and maintaining the processes of adding and subtracting decimals, examples are used as extensively as they are in textbooks. Just as when these processes with whole numbers were being learned, pupils are asked to prove that their answers to a few examples are correct. The common-fraction solution is the proof most commonly used.

In the procedure described, emphasis is placed on pupil-doing. Suggestions such as the solution by common fractions or diagrams are intended to bring out the relation between this new way of adding and the ways the pupil already knows. It should be noted that the illustrative problems used only tenths. Many texts introduce the addition of decimals through problems requiring the addition of dollars and cents. Such money problems for introductory work have two limitations. Hundredths being only a tenth as large as tenths are not so easily visualized as are tenths, and pupils have already learned to add and subtract dollars and cents without being aware of their decimal nature. The use of such situations to show the need for adding decimals, then, seems out of place.

The addition of decimals is very easy for most elementary-school children. In fact, as soon as the numbers are properly written, tenths under tenths, hundredths under hundredths, and so on, the process is identical with the addition of whole numbers. The rule, "Write the decimal points one under the other," is used by almost all teachers in teaching the addition of decimals. Occasionally, texts require the addition of such ragged decimals as $3.8 + 2.17 + 5.1 + 6.013$. Since there are few situations in life in which quantities are measured accurately to a tenth on one trial and a thousandth on another, there seems to be little practical use for the addition of ragged decimals. If measures are accurate to the nearest tenth, then the only fractions in the data will be tenths. There may, however, be some whole numbers in the data. To show that such whole numbers are accurate to the nearest tenth, a zero should be written in the tenths place. If whole numbers and decimals or ragged decimals are added, the sum is correct only to the measure that is common to all addends. Thus, there seems to be no sound reason for teaching pupils to add ragged decimals. Where such decimals occur in practice exercises, pupils should be taught to fill all vacant spaces with zeros. Those few instances

in which ragged decimals have to be added, as in the combining of data from different experiments, involve statistical principles that are beyond the field of elementary arithmetic.

In subtracting decimals from whole numbers a situation somewhat similar to the addition of ragged decimals is encountered. In the case of subtraction, however, the situation arises frequently in life and should therefore be taught. In teaching pupils to subtract in such situations ($7 - 3.18$) the writing of as many zeros in the minuend as there are places in the decimal of the subtrahend is recommended.

Multiplication and Division of Decimals

The same general classes and order of examples are used in the multiplication of decimals as were used in the multiplication of common fractions: first, a decimal multiplied by an integer; next an integer multiplied by a decimal. The procedure followed in teaching pupils how to multiply each class of examples is similar to that used in teaching other new processes. Problems are selected to illustrate each type; the procedure for multiplying is worked out from the related processes of adding decimals and multiplying by the common-fraction method; and finally examples are used to fix the process.

The development of a rule for placing the decimal point in the product is one of the main tasks in learning how to multiply decimals. In the development of this rule, extensive use is made of the common-fraction method of multiplying fractions. The following procedure is illustrative:

Assignment: From these examples try to make a rule for placing the decimal point in products.

(*a*) $\begin{array}{r} .4 \\ \underline{\times .6} \end{array} = \frac{6}{10} \times \frac{4}{10} = \frac{24}{100}$ Therefore, $\begin{array}{r} .4 \\ \underline{\times .6} \\ .24 \end{array}$

(b) $\begin{array}{r} .03 \\ \underline{\times\ 2} \end{array} = \frac{3}{100} \times \frac{2}{10} = \frac{6}{1000}$ Therefore, $\begin{array}{r} .03 \\ \underline{\times\ .2} \\ .006 \end{array}$

(c) $\begin{array}{r} .9 \\ \underline{\times\ 8\ 5} \end{array} = \frac{85}{10} \times \frac{9}{10} = \frac{765}{100} = 7.65$ Therefore, $\begin{array}{r} .9 \\ \underline{\times\ 8\ 5} \\ 7.65 \end{array}$

Test your rule on these examples. First multiply with decimals and then check by changing to common fractions and multiplying.

(a) $\begin{array}{r} 1 \\ \underline{\times\ .3} \end{array}$ (b) $\begin{array}{r} 5\ 2 \\ \underline{\times\ 4} \end{array}$ (c) $\begin{array}{r} 03 \\ \underline{\times\ .04} \end{array}$ (d) $\begin{array}{r} 10 \\ \underline{\times\ .02} \end{array}$ (e) $\begin{array}{r} 2.5 \\ \underline{\times\ 1.2} \end{array}$

The recommended order of presentation in teaching division of decimals differs markedly from the order followed in teaching other processes with both common and decimal fractions. The main classes of examples in order of presentation are as follows:

A. Mixed decimal numbers divided by whole numbers (6 5 ÷ 2)

B. Mixed decimal numbers divided by mixed decimal numbers (7.5 ÷ 2.5)

C. Decimals divided by decimals (8 ÷ .2)

D. Whole numbers divided by decimals (12 ÷ .3)

The use of a mixed number is recommended for the beginning work because many real-life problems are of that type, and because pupils can make a good judgment regarding the answer to such problems by considering only the whole number. This approximate answer is of great value in showing a need for a decimal point in the quotient. To illustrate, consider this problem: "After walking for three hours, one of the boys looked at his pedometer. It showed that they had walked 9.3 miles. What was the average rate per hour?" Few pupils would be willing to accept 31 miles as a reasonable answer to the question in that problem, and thus a need is created for determining where to place the decimal point.

The device of changing of decimal fractions to common fractions and then dividing is frequently used in determining the place for the decimal point. An example such as $9.3 \div 3$ then becomes $9\frac{3}{10} \div 3 = 9\frac{3}{10} \times \frac{1}{3} = 9\frac{3}{30} = 3\frac{3}{30} = 3\frac{1}{10} = 3.1$. After the correct quotients for several examples of division of a mixed decimal by a whole number have been found, a rule for placing the decimal point is formulated This rule, which states that the quotient decimal point is to be placed immediately above the dividend decimal point, will be found inapplicable in examples in which there is a decimal in the divisor.

The most common device for placing the quotient decimal when the divisor contains a decimal is the caret method. The divisor is changed to a whole number by moving the decimal point to the end of the divisor number. A caret is used to show the new placement of the decimal point. The decimal point in the dividend is then moved a corresponding number of places. Again a caret is used to indicate the new decimal point. The quotient decimal point is then placed above the caret of the dividend. The examples below illustrate the procedure.

$$(a)\ \begin{array}{r} 2 \\ 2.4_\wedge\overline{)4.8_\wedge} \end{array} \qquad (b)\ \begin{array}{r} 4. \\ .2_\wedge\overline{).8_\wedge} \end{array} \qquad (c)\ \begin{array}{r} 4\,0. \\ .3_\wedge\overline{)12.0_\wedge} \end{array}$$

The caret method has been criticized because of its mechanical aspects.[1] Since all the popular texts use this method and no markedly superior method has been proposed, there appears to be little chance that the method will be changed in the near future.

The use of decimals as a means of comparison, as in comparing $\frac{3}{4}$, $\frac{2}{3}$, and $\frac{1}{6}$, presupposes a knowledge of how to change common fractions to decimals. This process should not be taught until after division of all types of decimals has been taught. The

[1] Claude H Brown, "Some Thoughts on Placing the Decimal Point in Quotients," *The Mathematics Teacher*, *38* 78–80 (February, 1945), and H. Van Engen, "Some More Thoughts on Placing the Decimal Point in Quotients," *ibid.*, *38*. 243 (October, 1945).

general rule, "Place decimal point and one or two decimal zeros after the numerator and divide by the denominator," should be developed from study and experimentation with such known decimal common-fraction equivalents as $\frac{1}{10} = .1$, $\frac{1}{2} = .5$, and $\frac{1}{4} = .25$. As a part of the work of changing common fractions to decimals, some teachers advocate learning the decimal equivalents of these common fractions: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, and $\frac{7}{8}$.

After the study of the multiplication and division of decimals, some programs emphasize the three kinds of problems (case method) with decimals. There are perhaps better reasons for emphasizing the three categories with decimals than with common fractions. However, the same limitations that were pointed out in the discussion of this procedure with common fractions apply also to decimals.

The Status of Percentage in Present-Day Arithmetic Programs

Within recent years the teaching of percentage has been entirely eliminated from Grade Five, is practically eliminated from Grade Six, and in many programs only Cases I and II are now taught in Grade Seven. Case I problems involve finding the per cent of a number; Case II problems, the per cent that one number is of another; and Case III problems, the total number when only a per cent of the total is given. The reasons for delaying the teaching of percentage are many, but by far the most frequently mentioned are the difficulties children encounter in solving percentage problems, especially those classified as Case III. Other important reasons for delaying the teaching of percentage are those which support the general trend toward moving up the grade level of most topics. The lack of use that children have for computational percentage is also cited as an argument in favor of delay.

While the reasons for postponing the teaching of percentage appear to be sound and the changed grade placement has been generally accepted, the postponement procedure has created some serious difficulties for elementary-school children. Such statements as "35% of the coal mined is used to smelt iron," "20% of the corn crop was damaged by the freeze," and "More than 96% of the cases now fully recover," are unintelligible without an understanding of per cents. Fifth- and sixth-grade social studies, science, health, and current news make use of many statements similar to those mentioned. To meet the needs created by such uses of per cents, teachers in Grades Five and Six have tried to teach some meanings of per cent without teaching any of the computational procedures. No definite practice has been established, but the common-fraction equivalents of these per cents are usually taught. 10%, 20%, 25%, $33\frac{1}{3}$%, 50%, $66\frac{2}{3}$%, and 75%. While this practice is not ideal, the arithmetic program has to give some attention to the problem.

Since most percentage is taught above Grade Six, the topic is not treated as thoroughly in this volume as are other topics. In the sections that follow, the meaning of per cents, the case method, and other common methods of teaching are discussed briefly.

Developing the Meaning of Per Cents

The term *per cent* means hundredth or a hundredth part of a quantity. Five per cent of a quantity, then, means five hundredths of the quantity. If per cents mean hundredths or hundredth parts, what useful purpose do they serve? Why not say hundredths and thereby have only one word instead of two? The following uses of the term *per cent* will show its superiority over *hundredths*. (1) In the last 10 years the city has grown 200%. (2) The gain in weight was 100%. (3) The tax amounts to 1.25% of each sale. In none of these cases could

the term hundredths be used without modification. Another reason for using per cents rather than hundredths is that we experience greater difficulty in thinking and communicating with fractions than we do with whole numbers. The changing of hundredths to per cents is, then, very similar to the changing of yards to feet, feet to inches, pounds to ounces, and similar changes which we make in the field of measures. Instead of using three-fourths of a pound, we prefer to deal with twelve ounces. Per cents are then a special name for hundredths which we use in describing quantities. They are used as whole numbers without reference to their fractional aspects just as minutes and seconds are used without reference to their being fractional parts of hours.

One of the major uses of per cents is in making comparisons. Fractions are very useful in comparisons, if all fractions can be changed to fractions of the same denominator. Decimals are more useful than common fractions because all decimals can be readily changed to the same common denominator. Per cents are still more useful for comparison than are decimals, for not only do all per cents have the same denominator, but whole numbers and fractional parts of per cents can be included without difficulty. For example, 3.2%, 150.5%, and 86% are readily compared.

Since the meaning of per cents may be needed by elementary-school children before they have studied decimals, the common-fraction approach is probably the best one to use. The first teaching should be based on a list of statements taken from geographies, encyclopedias, news items, health reports, and the like. Statements such as these are typical: "66% of all the lead mined in 1938 was used in making storage batteries"; "The Mid-Continent field produced 20% of the oil in the U. S.", and "Germination is 90%." The assignment for the class would be somewhat as follows: "How can we show what 66%, 20%, and 90% mean?" To direct the thinking of the children such

questions as the following are useful: "Does it mean all? Does it mean more than a half? What part of all do you think it means?"

In explaining the meaning of 66%, 20%, and 90%, the fraction idea may be presented somewhat as follows: "Not all the lead was used for storage batteries. Let us suppose that all the lead mined was divided into one hundred parts; 66% would be 66 of those one hundred parts. Then, 66% means $\frac{66}{100}$, 20% means $\frac{20}{100}$, and 90% means $\frac{90}{100}$." Some teachers use dollars and cents to help children get the meaning of per cents. They explain in this manner. "If a dollar stands for all, then 66% of a dollar is equal to $\frac{66}{100}$ of a dollar or 66 cents."

The circle graph with a given per cent properly colored is an excellent device to use in helping children to grasp the idea of per cents. To represent 50%, half the circle would be colored and labeled both 50% and $\frac{1}{2}$. Ideally such a graph should present data on some class project In one class where lumber production was a topic studied in the social studies, 50%, 40%, and 10% were shown on the graph labeled "U. S. Lumber Production by Regions." On the graph were "Western States 50% = $\frac{1}{2}$," "Southern States 40% = $\frac{2}{5}$," "Central and Northeastern 10% = $\frac{1}{10}$."

Because the circle graph is difficult for children to construct, a bar graph is preferred by some teachers. The entire bar represents 100% and the desired per cents are marked by shading or coloring appropriate parts of the bar. Squares containing 100 small squares are another device frequently used to show the meaning of per cents. To show 20% on such a square, twenty of the small squares are shaded or colored. This device is frequently used in textbooks. In using such devices as the square, the bar, and the circle, not only should such common per cent equivalents as one-half and one-third be stressed, but the *hundredth* meaning of per cent should be emphasized frequently.

The changing of per cents to decimals and common fractions

is usually considered a part of teaching the meaning of per cents: $50\% = .50 = \frac{1}{2}$, $25\% = .25 = \frac{25}{100}$, and $\frac{1}{5} = .20 = 20\%$. From a study of examples of this type, the procedures and rules are developed for changing a per cent to a decimal and a decimal to a per cent. Adequate practice with all types of examples should then be provided. It is important that this practice include the changing to decimals of per cents greater than 100 and less than 1.

METHODS OF SOLVING PERCENTAGE PROBLEMS

Percentage problems are of the three kinds already referred to as: Case I, finding the per cent of a number; Case II, finding what per cent one number is of another; and Case III, finding a number from a given per cent.

The solution of Case I problems presents only one new thing to the pupil who has learned how to solve problems with decimals; that is, the changing of the per cent to a decimal. For example: "If 6% of a miner's car of coal is slate, what is the weight of slate in a car containing 4000 pounds?" The number question to be answered here is, "What is 6% of 4000?" Since 6% means .06, the number question can be changed to "What is .06 of 4000?" Thus, the percentage problem becomes a decimal-fraction problem. Then, to find the per cent of a number, the per cent is written as a decimal and the number multiplied by this decimal. In Case II problems the decimal fraction that one number is of another is first found and then the decimal is changed to per cent. In Case III problems the per cent is changed to a decimal and the part of the number given is divided by the decimal.

The procedure recommended for teaching pupils how to solve the three kinds of percentage problems varies from that used in introducing other processes. The initial assignment for the first kind of problem (Case I) illustrates the variation:

Assignment: In these problems you are to find the per cent of a number. First decide upon the essential number question needed to solve each problem; then change the per cent in that number question to a decimal. After that, you will know how to proceed.

Thus, instead of developing a new process in the usual manner (that is, by suggesting that the class work with known though longer processes), the teacher tells the pupil how to proceed. A period of development for percentage is omitted because there is essentially nothing new in the solution of percentage problems.

From the illustrative assignment it can be seen that the formulation of the essential number question to be answered is the important step in the solution of percentage problems. Here, as in the case of common-fraction problems, the process to be used in solving Case I problems is indicated by the essential question. A sample problem, with the essential number question, is provided below for each kind of problem.

Case I problem: "A cow produces on the average 60 pounds of milk a day. If 5% of the cow's milk is butterfat, how many pounds of butterfat does she produce per day?" The number question to be answered for the solution of this problem is, "What is 5% of 60?"

Case II problem: "Twelve of the 60 boxes of berries in a crate were ruined. What per cent of the boxes were ruined?" The number question needed for the solution of this problem is, "12 is what per cent of 60?"

Case III problem: "The profit of Company A was 6% of its total sales. If the profit of the company was $24,000, what were the total sales?" The number question to be answered in solving this problem is, "6% of what number is $24,000?"

When the per cent in the number question for the Case I problem above is changed to a decimal the question is, "What is .05 of 60?" or, "What is .05 × 60?" The arithmetical procedure to be used in answering the question is clearly indicated.

In solving the Case II problem the term "per cent" in the question, "12 is what per cent of 60?" is first changed to "decimal fraction." In the question, "12 is what decimal fraction of 60?" the process of division is definitely implied, although not indicated as clearly as is the multiplication in Case I problems.

In solving the Case III problem the per cent is first changed to a decimal. The question then becomes ".06 of what number is $24,000?" or, ".06 × what number is $24,000?" In this form the question is similar to "6 × what number is 24?" — a type of question used in solving one kind of whole-number problem. In answering this last number question, the 24 is divided by 6; in a like manner, division is used to solve the question, ".06 × what number is $24,000?" It should be recalled that many programs of instruction do not teach Case III of percentage until the eighth grade.

The Case Method of Solving Percentage Problems

Many texts now use what is known as the "case method" of solving percentage problems. By this method pupils are taught to identify the case or kind of percentage problem and then to use the solution suggested for that kind of problem. For example, if a problem is Case I, then the given per cent is changed to a decimal and the given number multiplied by the decimal. For Case II, the number asked about is divided by the other number and the resulting decimal changed to a per cent. For Case III problems, the given number or part is divided by the given per cent expressed as a decimal.

In the chapter on common fractions the limitations of the case method as applied to the solution of fraction problems were discussed. Since those same limitations may apply to percentage problems, they will be repeated here.

1. In using the case method the pupil centers his attention on identifying the case and therefore does not give enough atten-

tion to an over-all study of the problem in order thoroughly to understand it.

2. In learning to distinguish the three kinds of problems, the pupil has to learn something extra; he has to use an extra step between the consideration of facts of the problem and the solution. According to some teachers, the skill required to identify problems is not essential to the solution of percentage problems.

3. The case method tends to become mechanical and lacking in meaning. The popularity of the case method for percentage in textbooks seems to indicate that the limitations listed above are not so serious as they might appear at a first reading.

Unitary Analysis and Equation Methods

Case III problems in percentage, finding a number from a given per cent, have been by far the most difficult for pupils to master. In addition to the two methods of solution already listed, two other methods are in common use. These are the unitary analysis method and the equation method. These two methods are illustrated in the solution of this problem:

Problem: At a seed corn processing plant the field-ripe corn is dried and graded. The losses due to these two processes leave only 80% of the original weight to be sold as seed. At that rate how many pounds of field-ripe corn are needed to produce one bushel of seed weighing 56 pounds?

The solution of this problem by the unitary analysis method is as follows:

$$
\begin{aligned}
80\% \text{ of the weight} &= 56 \text{ lb.} \\
1\% \quad \text{"} \quad \text{"} \quad \text{"} \quad &= \tfrac{56}{80} = .7 \text{ lb.} \\
100\% \quad \text{"} \quad \text{"} \quad \text{"} \quad &= 100 \times .7 = 70 \text{ lb.}
\end{aligned}
$$

The solution of the same problem by the equation method is shown below:

$$
\begin{aligned}
&80\% \text{ of what number is } 56? \\
80\% \times n &= 56 \\
.80 \times n &= 56 \\
n &= \frac{56}{.80} = 70
\end{aligned}
$$

Although the unitary analysis method is advocated only for Case III percentage problems, the equation method is suggested by its supporters for use in solving Case I and Case II problems also. For a Case I situation such as "What is 12% of 48?" the equation $n = .12 \times 48$ is used. For a Case II situation such as "8 is what per cent of 20?" the equation "$8 = n \times 20$" is used. The chief limitation to the use of the equation method in percentage as in fractions is the fact that pupils are not familiar with equations. In other words, they must first be taught how to use the equation before that method can be successfully employed in the solution of percentage problems. Some teachers who have tried the equation method report that teaching pupils to deal with equations is not a serious obstacle.

STUDY QUESTIONS

1. One reason for teaching common fractions before decimals is that they provide essential background for decimals. What other reason is there? *(1)* Common fractions are easier to compute with. *(2)* Decimal notation is too difficult to master. *(3)* The basic idea of fractions is developed more easily with common fractions. *(4)* N.

2. In common fractions the largest unit fraction is the one that occurs most frequently. Is that also true of decimal fractions? *(1)* Yes. *(2)* No.

3. Why is decimal notation as used in money not considered the best setting for introductory work with decimals? *(1)* Because only a small per cent of the children have had experience with money *(2)* Because the hundredth as used in money is not the same as the hundredth in decimals. *(3)* Because you do not use the decimal aspect of money when computation with money is done. *(4)* N.

4. What is the most important reason for opposing the use of ragged decimals in arithmetic teaching? *(1)* Elementary-school pupils are too immature to grasp the meaning of signifi-

cant figures. (*2*) Ragged decimals are too difficult to compute with. (*3*) Ragged decimals never occur in problems that concern elementary-school pupils. (*4*) N.

5. Decimals are more useful for purposes of comparison than are common fractions. Why? (*1*) Because they are easier to read (*2*) Because they can be readily changed from one denomination to another. (*3*) Because they are easier to visualize (*4*) N.

6. In showing children the value of decimals over common fractions ease of writing is one of the arguments used. What is another? (*1*) Decimals present the meaning more clearly. (*2*) Decimals make for neatness of presentation. (*3*) Decimals are more easily rounded. (*4*) N

7. Why are the addition and subtraction of decimals either taught simultaneously or within a very short period of time? (*1*) Because the two are closely related and derive meaning from each other. (*2*) Because each process is easily mastered and therefore a long wait is not needed. (*3*) Because the two are interdependent. (*4*) N.

8. In multiplying decimals what important new thing does the child learn? (*1*) To keep decimal points in line. (*2*) To have as many decimal places in the multiplicand as in the multiplier. (*3*) That the size of decimals has little effect on the difficulty of the multiplication to be performed. (*4*) N

9. Which of these division of decimal situations is the most difficult? (*1*) $4 \div .2$. (*2*) $.2 \div 4$. (*3*) $2.4 \div 1.2$. (*4*) $.4 \div .2$.

10. In connection with what process is the caret frequently used? (*1*) Changing decimals to common fractions. (*2*) Division of decimals (*3*) Multiplication of decimals. (*4*) N.

11. Since per cents mean hundredths, what value over decimals is there in the use of per cents? (*1*) There is none. (*2*) In per cents fractional and multiple parts can be shown without changing the nomenclature. (*3*) Per cents are easier to write than hundredths. (*4*) N.

12. Problems of which of the three cases of percentage occur least frequently? (*1*) Case I. (*2*) Case II. (*3*) Case III.

13. The unitary analysis method of solution is best suited for which case of problem? (*1*) Case I. (*2*) Case II. (*3*) Case III.

14. On what grounds do some teachers oppose the use of the case method of solving percentage problems? (*1*) It is too long (*2*) It leads to extreme emphasis on computation. (*3*) It leads to consideration of minor rather than major aspects of the problem. (*4*) N.

15. For what reason do some teachers prefer the use of the bar graph rather than the circle graph in showing the meaning of per cents? (*1*) The bar expresses more clearly the whole. (*2*) The bar is more easily constructed (*3*) It is easier to make comparisons with two bar graphs. (*4*) N.

11

The Course of Study in Arithmetic

Courses of Study in Use

The textbook is the course of study in the great majority of schools today. The local course of study in most cases merely designates the book to be used and the areas of arithmetic to receive attention during the year. The use of the textbook as a guide to instruction in arithmetic has a great deal of merit. With the possible exception of spelling, there is no other elementary-school subject where the text can be used so successfully as the course of study. Certainly a good series of textbooks with the accompanying teachers' manuals is a better guide to instruction than the course of study that teachers' committees can prepare in one or two years. The authors of texts are usually acquainted with the chief issues in arithmetical instruction, and even if an individual author knows little of the field, by intensive study of the texts already in use he can learn the most important things to include in a text.

The foregoing discussion is not an attempt to justify the use of a textbook as the single guide to instruction. Even though that procedure has some good points in its favor, it also has some very definite limitations: (1) Textbooks are made to sell in many places. Hence, in their attempts to please all prospective customers, authors have been forced to include more material

than is necessary for most school systems. (2) Textbook problems must perforce deal with general quantitative situations; consequently, any school program following the text alone will have no problems which arise in the everyday activities of the pupils. (3) Textbooks cannot express the point of view of the local school with regard to the arithmetic of science, geography, or spelling to be included in the instructional program. (4) Textbooks usually provide alternative methods where several methods of presentation are in existence; for example, in the teaching of subtraction both additive and take-away methods are usually presented. The local system must select the one to be used. (5) The textbook cannot be of much service in providing a general survey achievement test.

In addition to these limitations of texts as courses of study, the fact should be recognized that not a single text on the market today makes use of the method of instruction described in this book. Nor is much place given to oral arithmetic, approximation, the use of diagrams, or the history of numbers. If the practices advocated in this book are to be used, it is therefore necessary to build a course of study which the teacher can use as a guide.

At the present time many city school systems and state departments of public instruction have excellent courses of study, or guides to instruction in arithmetic. Because the state department materials are planned for use with many different textbooks, they are often rather general, but they contain good suggestions for grade placement, measurement, motivation, specific suggestions on method, and the like. City system materials, designed for one system only, are able to make specific suggestions. Where a text is used, the course of study is often a guide for the use of text materials and also contains suggestions for enriching and motivating instruction. By emphasizing certain content (such as the reading of Roman numbers and the history of arithmetic), by recommending the omission of

some content (such as proportion), and by placing the teaching of certain topics (such as percentage) in a grade above common practice, courses of study have done much to change the content of arithmetic textbooks. For a list of courses of study, consult the reference list at the close of this chapter, page 324.

Building the Course of Study

So much has been written in the last ten years on construction of the course of study (usually the broader term *curriculum* is used) that discussion of the major issues in that field seems unnecessary here. The revision of arithmetic courses of study or curriculum guides is seldom undertaken as a separate task, but is usually a part of a general or over-all curriculum-revision program. The common procedure is the appointment of a subcommittee on arithmetic, which works under the general direction of a central curriculum committee. Not infrequently arithmetic is combined with other subjects to form such divisions as "The Skills Area" or "The Science-Mathematics Area." From a study of the hundreds of courses of study, guides to instruction, handbooks, and arithmetic curriculums which have been built in the last fifteen years, no one generally accepted procedure can be determined.

One of the outstanding characteristics of curriculum-revision programs has been the attempt to secure the goodwill and aid of the teachers who are to use the course. Usually this has been done by appointing a course of study committee that is largely made up of teachers, a procedure sometimes justified as being democratic. If goodwill, genuine contributions, and the interests of democratic practices can best be achieved by turning the work over to teachers, then that procedure should be followed. It must be recognized, however, that teachers are usually too busy to make the careful study of the methods of instruction and the facts of arithmetic that are prerequisite to genuine contributions to the curriculum. Teachers must first

be given an opportunity to learn before they are asked to make contributions. Instead, then, of asking teachers to assume the major responsibility, some person who has the time and knowledge should prepare a tentative plan or course of study which can be used as a basis of discussion. Following such discussion, a revised plan can be presented for try-out.

Teacher goodwill and teacher contributions will be gained through the discussion and actual try-out of plans and procedures. Any attempt on the part of outsiders (even curricular experts) to tell teachers exactly what to do is almost certain to result in inferior teaching. On the other hand, if the problem is approached in the spirit of inquiry, with both curriculum director and teacher searching for the best material to present and the best method of presenting it, superior teaching may be expected. Even after teachers have tried out a plan and discussed the various issues in curriculum meetings, it is still a big job to organize and put the course of study in final form — a job perhaps better handled by a person other than a classroom teacher. Of course, if teachers have the time and the ability to do the work, there is no objection to having them prepare the course of study.

Significant Parts of the Course of Study

Courses of study in arithmetic are of two general types: one designed for use in connection with textbooks, the other, for use without a text. To be successful, however, the second type requires the development of a large amount of problems and examples which in a sense are textbooks. The course of study designed for use without a text has been more successful at the primary level than at the intermediate and upper grade levels. It should be obvious that the content of the course of study planned for use with a textbook will vary markedly from that of a program not based on a text. The former will provide little, the latter a great deal, in the way of sample instructional

material. But there are a number of things common to both types of course which are important enough to warrant special attention and which are enumerated below as "Significant Parts of the Course of Study," listed in the approximate order in which they appear in the course of study: (1) statement of philosophy with listing of specific major purposes; (2) chief features of the method of teaching to be followed; (3) the place of drill in the teaching procedure; (4) special instructional devices, aids, and procedures that can be used in teaching each major area of arithmetic, (5) the grade placement of topics; and (6) illustrative lessons. Each of these six topics will be discussed briefly.

1. *Philosophy and Guiding Principles*

The purposes and principles that the author suggests for a course of study would be similar to those given in Chapter 2 (see pages 23 ff.). Another statement of those purposes and principles here would be needlessly repetitive. The statement of philosophy, whether in the form of purposes or in an expository discourse, should be relatively brief. Teachers who are to use the course of study know that little if any direct help is to be drawn from reading statements of purposes. Since such statements are not considered valuable by teachers — are, in fact, taken rather lightly — the section on philosophy should include a brief exposition on the place and importance of arithmetic in the whole instructional program. An example of such an exposition is given below.

This course of study is based on the premise that arithmetic is to be an important part of the total instructional program for children. The term *important part* is not intended to convey the idea that arithmetic is to become the center or even one of the major centers around which other school work of a child is to be built. It is contended, however, that the field of arithmetic is important enough in the child's education to warrant being

given a place of its own in the school program. While arithmetic can and often should be related to other fields of interest, the arithmetic program is not to be "tied to," nor "to grow out of," or in any other way be subordinate to other centers of interest. In other words, some of the children's time can be spent as profitably in the direct study of arithmetic as in the study of any other area.[1]

2. *Major Characteristics of the Method of Teaching to Be Used*

The importance of identifying major characteristics of teaching methods is shown by the questions that were asked by a new teacher in a school system. (*a*) "In teaching the initial phases of new topics, such as multiplication, am I to follow the method of the book, or may I use my own modification of that method, or do you have a special method to follow?" (*b*) "Can I afford to take the time to develop a thorough understanding before we start to practice?" (*c*) "Am I to use actual demonstration with objects in proving, or is checking sufficient?" (*d*) "Should I have my fourth-grade pupils work the explanatory lessons in the book, or is reading and discussion sufficient?" (*e*) "I find no flash cards in my room. Is the administration opposed to their use?"

While some of these questions are concerned with minor details of method, others involve fundamental issues. The most important characteristics of method to be identified are those concerning initial instruction. If the teacher is expected to follow a method of teaching that emphasizes telling and demonstrating, then the course of study should state just that. If the teacher is expected to use a method that emphasizes experiences which make for the development of facts and processes, then that method should be identified in the course of study. Since most courses of study will be designed for use with textbooks, the methods particularly identified will be those which

[1] *Arithmetic in the University Elementary School* (Iowa City: State University of Iowa).

differ from the ones recommended in the textbook, and special attention will be called to those textbook methods which are to be emphasized.

3. *The Place of Drill in the Teaching Procedure*

Since drill is an established part of the various teaching procedures, it might be included under the category of "Characteristics of the Method of Teaching." Primarily because drill is so often misunderstood, separate treatment in the course of study is recommended. The author recommends that the section of the course of study dealing with drill include a statement explaining the necessity for drill and the four conditions of a good drill program. For further discussion of drill, see Chapter 14, page 381. Since teachers often find that the amount of drill in the textbooks is inadequate, the course of study should list special sources of drill material, such as the mimeographed exercises that are often prepared by supervisors and principals.

4. *Special Instructional Devices that Can Be Used in Teaching Major Areas of Arithmetic*

A number of devices and special materials can make the teaching of various phases of arithmetic much easier for the teacher. These should be listed in the course of study because teachers may be unfamiliar with them or may have been instructed in other schools not to use such procedures. For example, counting on the fingers, a procedure banned in many school systems, is recommended by the author.

In teaching counting and the idea of tens as applied to numeration, it will be advantageous to use: (*a*) the fingers; (*b*) large numbers of small objects such as beans and grains of corn; (*c*) number cards showing numerals and dots; (*d*) tens and ones blocks, (*e*) a number chart; (*f*) the tens square; (*g*) the calendar; and (*h*) the abacus.

In teaching place value, the abacus and tens blocks, as well as bundles of sticks, will be assets.

In teaching the foundation for the addition and subtraction facts and in teaching the basic combinations, the following materials will be useful: (*a*) dominoes; (*b*) number cards containing both numerals and dots; (*c*) tens and ones blocks; (*d*) a quantity of small objects to use in proving and demonstrating the fundamental nature of the two processes; (*e*) the abacus; and (*f*) flash cards.

Useful devices for teaching carrying and borrowing are the tens blocks, the abacus, and bundles of sticks. For teaching multiplication and division also, the tens and ones blocks will be very beneficial. In addition, flash cards and special tables can be used to advantage.

For teaching the basic ideas of linear measurement, sticks or other unstandardized measures of length are needed. Of course, as the instruction progresses the standard measures will be required. In teaching measures of weight, a scale is needed. In teaching measurement of area, unstandardized as well as standard unit squares will be very helpful.

In teaching fractions, special circles and parts of circles representing the most common fractions are needed

5. *The Grade Placement of Topics*

This section of the course of study consists of an outline of the content to be taught in each grade. Where textbooks are used, the course of study need list only the deviations from the content given in the textbooks. Many courses of study suggest also the amount of time to be allotted to the teaching of major areas of content. For example, one course of study for the third grade suggests that the first six weeks be used for review of number concepts, counting, and basic addition and subtraction facts, and that the next nine weeks be devoted to teaching addition and subtraction of two- and three-figure numbers.

6. *Illustrative Lessons*

The illustrative lesson or description of actual teaching procedures used in a classroom has been one of the effective ways employed by supervisors in introducing new methods. Teachers quickly get the point of a new method when they see how it works in the classroom. Furthermore, since descriptions of lessons merely narrate what has been done, teachers do not feel that new procedures are being dictated to them. To test the possibilities of descriptive lessons as a means of introducing something new, read lesson 2 in Chapter 13 (page 350).

In view of the fact that illustrative lessons are an effective means of acquainting teachers with new procedures, probably as much space should be given to this section of the course of study as to any other section. Although there is no evidence to support the recommendation, the author suggests at least two illustrative lessons for every grade. For examples of illustrative lessons, consult Chapters 3, 4, 13, and 14.

In addition to the six significant parts, other topics might well be treated in separate sections of the course of study, for example, oral arithmetic, testing, and outstanding characteristics of the number system. Each of these areas of arithmetic is discussed elsewhere in this book. (See Chapters 1, 2, and 12.)

Selection of Curriculum Materials

1. *Theory of Social Utility Method of Selection*

In the exposition of the six significant parts of the course of study there is no statement as to how the content of arithmetic is to be selected. In Chapter 2 it was stated that the ultimate value of any subject must rest upon its contribution to life. Out of such a pragmatic doctrine has arisen the theory of social utility which has been extensively used as a guide in the selection of curriculum content. Briefly, this theory holds that there shall be a one-to-one correspondence between the things

taught in school and the things used in life outside the school. In other words, the arithmetic of the school should be the same as that used in life.

The theory of social utility has been most successfully applied in the field of spelling. The words most needed in writing are now known and taught in the spelling program of our schools. The location of these words was no easy task, but certainly a far simpler task than that confronting the researcher who tries to determine the phases of arithmetic most used in life outside the school. In spelling, a record is made every time a word is written, but in arithmetic there are many instances where no record is made when a number is used. For example, a person who sees something which he desires in a store window does not get out pencil and paper and do any exact subtraction in deciding whether or not he can afford to buy it. Again, no computation of any kind is performed when one reads that the height of a poinsettia in Florida is twenty-five feet, yet number is used in the statement, and that number helps the reader to form a good idea of the size of the poinsettia. This use of numbers in reading is perhaps one of the most frequent and most important applications of number. Numbers are essential also to profitable thinking about almost all problems, whether they be in agriculture, sociology, or science.

Failure to recognize that the pencil-and-paper records of number are only part of man's use of arithmetic is responsible for the poor quality of most of the research that has had as its goal the identification of the uses of arithmetic. Because they have been too narrow in their approach to the problem, these research projects have been of little value. The findings, furthermore, have been interpreted with a narrow conception of the true part that numbers play in life. For example, because research studies have shown that the work with fractions is primarily concerned with $\frac{1}{2}$'s, $\frac{1}{3}$'s, and $\frac{1}{4}$'s, the recommendation has been made that all work with fractions deal only with these

fractions rather than with a generalization that will apply to all fractions.

This same narrow outlook has led to the extreme position, now so prevalent in arithmetic programs, regarding the teaching of the basic facts in the fundamental processes. The usage studies have shown that almost all the uses of number by the ordinary man have to do with problem-solving. Analysis of the problems has shown that the basic facts are needed for the solution of problems. Following those findings and conclusions, arithmetic programs have set out to teach the facts so that children can solve problems. Problems of the type that are most frequently encountered in life are then provided as the final step in teaching.

This brief description of the procedure based on utilitarian studies makes the procedure sound better than it really is. With the narrowly conceived and narrowly interpreted usage studies as guides, there is no place for understanding of the number system, no place for seeing the nature of processes, and no place for seeing that one certain way of describing a quantity is better than others. In fact, only one way of presentation is considered the ideal way of teaching. The interrelationship between facts and processes, one of the major characteristics of an efficient use of numbers, is entirely neglected. Use seems to be the only purpose for teaching arithmetic. For a critical analysis of this "tool" view of subject matter, see Chapter I of the *Third Yearbook* of the National Council of Teachers of Mathematics.[1]

Among the further shortcomings of the narrow application of the social utility theory, particular mention should be made of the fact that these studies have eliminated from the curriculum many important areas of arithmetic, such as ratio and proportion. (The almost universal use of ratio is well shown

[1] *Third Yearbook*, National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1929), chap. I.

by the betting odds that are always quoted on important elections, prize fights, and football games.) The large amount of daily life arithmetic that is of the unwritten type was pointed out in Chapter 2. Yet a recent book, which claims to advocate an arithmetic program based on usage studies, makes no mention at all of oral or unwritten arithmetic.

2. Logical Analysis Method of Selection

From the preceding section it is reasonable to conclude that the research studies based on the theory of social utility have not yet produced acceptable data for the selection of content for an arithmetic program of the type referred to in this book. In the absence of complete data on the uses of number, a logical analysis of the mathematical system as it applies to life situations has served as a guide for the selection of arithmetic content. Both the narrow utilitarians and those who have made a logical analysis of the uses of number have reached the conclusion that number is a valuable tool for those who know how to employ it. The "utilitarians" have seized upon one of the most obvious of these uses, problem-solving, as the thing to teach The logical "analysts" have decided that the problem-solving use as well as other number uses can be taught by teaching the outstanding features of the number system. By one scheme, facts and processes are taught in order that children may learn to solve problems. In the other, problems are used to illustrate the mathematical facts, processes, and relations that form the framework of the number system. To the utilitarians no educational value, other than the provision of a useful tool, results from the teaching of arithmetic. To the logical analysts, as interpreted in this book, there is educational value in the learning of the number system. In addition, the child acquires the possession of a valuable tool.

Other differences in the content of the program, besides those mentioned above, will result if the logical analysts rather than

the utilitarians are followed. These differences occur because certain features are included in the program by the analysts which are omitted by the utilitarians. A list of these features will be almost the same as a review of the outstanding features of the arithmetic program described in this book. They are repeated here to emphasize the difference between this program and that built upon a narrow interpretation of the theory of social utility. The list is as follows: (1) Unwritten arithmetic is given a definite place in the program. (2) Approximation is taught and practiced in the arithmetic program. (3) Special emphasis is placed on developing the idea that tens, hundreds, and other collections are handled just as ones are handled. (4) Much of the teaching is directed toward the development of relations within the number system. (5) Many associations and facts about a list of standard reference points are developed. (6) Children are encouraged to make judgments in situations in which common measures are used. (7) The aim of much of the arithmetical instruction is to show how number makes for simplicity of thought. For example, instead of trying to think of the total population of each of two large cities, it is more convenient to express the relation between the two figures by saying that one is one and one-half or two times as large as the other. (8) History of number is included as a part of arithmetic.

3. *A Broader Interpretation of Social Utility Needed*

All eight points listed above should be included in the arithmetic program, because it is believed that each of them could correspond to an important use of number in life. If that belief should be supported by facts, the program would be in agreement with the theory of social utility. Thus, instead of opposing the theory of social utility, the writer merely asks for a broader application of that theory in the field of arithmetic. Much research needs to be done; and, to be of value to the curric-

ulum-maker, the research must be guided by a broader view of the role of numbers than has been exhibited in many studies. Some of the studies already completed are of value if the findings are interpreted in the light of the small area of life with which the research dealt. Contrary to the situation for spelling, however, the findings of usage studies in arithmetic can never assume the all-dominant role in the selection of curricular content. The number system itself is too powerful a factor to permit that to happen. In spelling there is no real system to make the task easier. But since number is made easy through the use of a system, that system should have much influence in the selection of teaching material.

In spite of the fact that usage studies will not be the sole guide in the selection of curriculum content, a plea is made for more careful studies of the uses of number. Not only would such studies be of value in choosing content, but they would be of value in the determination of teaching procedures.

STUDY QUESTIONS

1. What is the main reason for putting teachers on course of study committees? (*1*) They know better than anyone else what arithmetic should be taught to children. (*2*) They have in their possession long lists of tried procedures. (*3*) They know best how to tell other teachers what should be done. (*4*) N.

2. Why is the arithmetic textbook not a good course of study? (*1*) Because it includes too much material. (*2*) Because it does not include enough material. (*3*) Because the topics are not graded well enough. (*4*) N.

3. Why do teachers seldom make outstanding contributions to the content of courses of study in arithmetic? (*1*) They serve on curriculum committees so infrequently. (*2*) They are usually overshadowed by administrative officers on the committee. (*3*) They do not have the time to make a careful study of the field of arithmetic. (*4*) N.

4. Why do teachers pay little attention to statements of purpose in courses of study? (*1*) Because they are not interested in goals or objectives. (*2*) Because these statements are of little help in the actual teaching process. (*3*) Because the teachers do not know how to interpret the statements of purposes. (*4*) N.

5. What special advantage is there in suggesting new procedures through illustrative lessons? (*1*) If it is in lesson form teachers know it is practical and not just theory. (*2*) In lessons there is no need for discussion of the pros and cons, and therefore the lesson presentation is very short. (*3*) The lesson tells the teacher every step. (*4*) N.

6. Why is it so difficult to find out what arithmetic is used in life outside the school as a means of selecting curriculum content? (*1*) Few people use arithmetic effectively. (*2*) In many uses of arithmetic no record is made. (*3*) Arithmetic is too complex to permit easy identification of its uses. (*4*) N.

7. For what reason should the course of study include a special section or statement on drill? (*1*) Because few teachers know how to use drill in instruction. (*2*) Because the textbook in use may not contain any drill material. (*3*) Because there is much confusion concerning the place of drill in arithmetic. (*4*) N.

8. In addition to the fact that it is difficult to gather the data in usage studies which are broad enough to secure a good picture of the arithmetic of life, what other limitation of this method of identifying the content of arithmetic must be recognized? (*1*) Arithmetic is a system, and therefore usage is not necessarily an indication of importance. (*2*) Usage is always changing. (*3*) Frequently the practices used outside of school are incorrect, as, for example, in the use of "and" in reading numbers.

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12

Testing in Arithmetic

The Status of Non-Standardized Testing

In America, tests or examinations in arithmetic are practically as old as arithmetical instruction. In early college-entrance examinations, teachers' examinations, local, county, and state examinations, the arithmetic section was usually quite extensive. Besides these rather formal examinations, much of the instructional time has always been taken up with testing. Today, every standardized achievement-test battery has an arithmetic section, and every instructional program in arithmetic is marked by the large proportion of the learning time that is given to testing.

Tests in arithmetic, like other subject-matter tests, may be conveniently classified under two categories: those that are teacher-made, and those that afford a general survey of achievement.

Teacher-made tests usually have a definite place in the instructional procedures. Such tests are commonly passed back to the children and become a part of a corrective or remedial program. Some of these teacher-made tests are also used to determine achievement, but this use does not interfere with the remedial function.

General survey achievement tests may be either standardized tests or merely tests set up by the local school, the local system,

or the state. These tests are not part of the instructional program, but are used primarily to evaluate instruction or achievement. In some cases the scores are used to determine academic ranking and promotion to the next grade.

In classes where a textbook is used, other opportunities for testing are provided in the form of self-testing drills, quick drills, review exercises, and assignments which require the solution of problems and examples without any other purpose than merely to find out whether or not the children can work them.

Testing followed by brief explanations seems to be the most popular of all instructional procedures in arithmetic. Observation of a week's work in many schools will lead to the conclusion that more than half the time is spent in testing. Even though testing is admittedly an integral part of the instructional procedures, it should be recognized that under such conditions over one-half the time is being spent in "finding out what the children know." Consequently, the time left for other instructional procedures is too short. To insure that instructional procedures other than testing receive the proper amount of time, not more than fifteen per cent of the total time should be devoted to testing.

In spite of the fact that much time is given to testing in arithmetic, most of the tests are rather mediocre in quality. The achievement score or rating of the child on most of the teacher-made tests and on the general survey tests is based almost entirely upon the child's ability to secure accepted answers to problems and examples. Such tests fail to measure abilities in other important areas, and place undue emphasis on rapid answer-getting.

Teacher-made tests, used as a part of regular instruction, are usually given after some time has been allotted to study in a specific area. Often the test is used primarily to motivate the work by pointing to the children's failure to do certain examples or problems. Such use is commendable, but can be easily over-

done. Tests covering specific areas of arithmetic are often misleading, because children may make fairly high scores even when their understanding and ability in the area are mediocre. These spuriously high scores probably result from the fact that the test deals with only one thing; consequently, after working one problem, the child is just copying. Even in working the first problem of a test, it may be considered that a child is copying if he merely does what he has been doing in the arithmetic lessons immediately preceding the test lesson Since almost everyone can copy far beyond his ability to produce, the scores made on a test on a specific area just studied can easily misrepresent the true achievement of the child. To avoid this limitation of tests, teachers should include items that pertain to areas of arithmetic previously studied.

Suggested Improvement

To avoid the criticism that tests deal only with getting correct numerical answers, there should be included some items which test other abilities, such as the use of diagrams as a means of solving numerical situations, the recognition of approximate relations, and the making of judgments. The following are examples of such items:

1. Show with a diagram that $\frac{1}{4}$ of $\frac{1}{3}$ is $\frac{1}{12}$.
2. If H received 127,321 votes and M received 251,108 votes, how could you express with small numbers the relative number of votes received by the two?
3. About how far is it from the floor to the picture on the wall?
4. Is $25 a reasonable price to pay for a pair of shoes?

Discussion with the children of the purposes and make-up of tests should be a part of the test procedure. Such discussion might be initiated by a question like the following: "When we have so little time in school, why is part of that time used in

taking tests like the one you took yesterday?" Children usually give the main reasons for taking tests. In their language the reasons are: (1) So that the superintendent (principal, teacher) can find out what we know. (2) To learn where we need special help. (3) To grade us. (4) To find out what I need to practice on. (5) To find out what I need to study. (6) To find out who is the best in the class. (7) To find out who needs to go to work.

Improvement of teacher-pupil relations and of class spirit or morale will result from a frank discussion of these reasons with the pupils, especially if the value of each reason is understood by the teacher and if she directs the discussion so that children will be able to see the value. Children can see that even reason 1 has value when they realize that the superintendent or teacher uses the information in selecting new materials of instruction or instructional procedures. For example, if a teacher can show through a test that the children have only a hazy idea of the size of an acre, a real motive for marking off an acre on the school grounds has been provided, and therefore it should be easy to obtain the permission of the administration for such a project. Special effort should be made to eliminate the type of thinking that gives rise to reasons 6 and 7, which are indicative of the ignorance of pupils about the true purpose of so common an instructional procedure as testing. Instead of being permitted to look upon high test scores as a notice of dismissal from further work, the children should be led to see that an indication of proficiency in a certain area points the way toward the undertaking of more difficult tasks. The crux of class discussion of tests is the idea that children should know something about the learning procedures in which they engage. A major part of their time is spent in school, yet few of them can give a clear-cut, logical explanation for many of their school activities.

Oral exercises as discussed on pages 107 and 146 are frequently a form of testing. When the testing aspect of oral work is

combined with the fact that such exercises force children to deal with number in the manner that out-of-school life frequently demands, these exercises certainly have much to commend them. Since this part of arithmetic is discussed on pages 52–55, further elaboration will not be made here. It is sufficient to state that oral exercises comprise an important part of teacher-made tests in arithmetical instruction.

The testing programs provided by textbook courses are usually of a superior type. The content of the tests is well distributed, items are graded as to difficulty, and the tests follow soon after the presentation of new phases of subject matter. The standard rating scales that accompany these tests are frequently used to indicate success or failure to the student. While it has been demonstrated that growth graphs and similar means of informing the student of progress make for better learning, it is questionable whether such extrinsic motivating factors can be of much value over a long period of time. Furthermore, the norms used in growth graphs are often of doubtful validity. Other more serious criticisms of the intensive testing programs of most textbooks are the large amount of time given to this phase of instruction and the premium placed on speed in computing. In one school system, every Monday arithmetic period was used for giving a test, and every Tuesday period for correcting and working items missed on the test. Thus forty per cent of the total arithmetic time was spent on test work. An equally serious limitation of textbook testing programs is the premium that most of them place on speed. If pupils are to secure good marks on the tests they cannot spend time checking or working on items that are not immediately perceived. Yet to fulfill its aim arithmetic ought to stress both checking and reasoning. This condemnation of the premium placed on the speed factor in tests may appear, at first thought, to be inconsistent with the recommendation that timed exercises be used to show the pupil the need for learning basic facts and fundamental processes.

(See Chapters 2, 5, and 6.) The timed exercises employed in teaching basic facts and processes, however, were designed to show that indirect and cumbersome methods were responsible for the lack of speed. It is the evaluation of various ways of getting answers which follows the timed exercises that gives point to such exercises. If the speed factor in all tests served a similar function, there would be less criticism of that factor. Textbook tests, like teacher-made tests, also suffer from the fact that practically all items require exact answers.

Inventory Tests

Inventory tests in arithmetic are of two types. One refers to an appraisal or inventory of skills essential to the learning of new processes. The other refers to a type of survey test which is given at the beginning of the year and which is designed to give the teacher a picture of the arithmetical achievement of the child. Since arithmetic is no longer thought of as a distinct or rigid step-by-step learning process, the first of these two types of inventory tests has become relatively unimportant and its use at present is quite limited. The second type of test is not used widely either, but it is considered a desirable testing procedure because it gives teacher and child a chance to see what type of review work is needed. Since the test is designed to survey total arithmetical achievement, it is obvious that the test must consist of something more than problems and examples. Naturally, the content of these inventory tests from first to sixth grade will show much variation. Sample items from a first-grade test, a third-grade test, and a sixth-grade test are given here.

First Grade

Tell the child you want to see how well he can count and how far he can count. If necessary, start him off on counting by saying, "1, 2, 3."

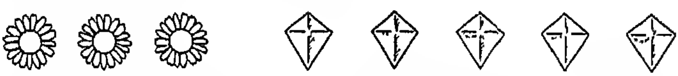
Place ten splints before each child. Ask him to count them for you. Have him point to each one as he counts.

Arrange objects in groups, such as two pencils, four blocks, seven marbles, ten crayons, and so on. Ask the child to point to the group with four objects; two objects; seven objects; ten objects.

Using pictures, say, "Put a mark over the picture of three kittens. Show me the picture of five birds Point to the picture of seven birds. Which picture has ten birds?"

Show two objects. (Any other number may be used.) Say to the child, "Bring me two blocks. Make four apples on the blackboard." (In each instance the teacher gives the name of the group she desires the child to select.)

Using work sheets or illustrations on the blackboard similar to the following, the teacher may say, "Put stems on three flowers, or put tails on five kites."



Show the child several blocks of various sizes. Have him point to the biggest block; the littlest block. Hold up one block and say, "This is a big block, show me a bigger block. Show me the biggest block."

Give one child two blocks. Give another five blocks. Ask, "Who has the more blocks?"

Third Grade

Write the number two hundred sixty-one.

Make ninety marks on your paper. Show how you would count these marks by tens by drawing rings around the right number of marks.

How much are 7 and 8?

How much are 9 and 3?

How many are 9 take away 4?

Write the number meaning 8 tens and 7 ones.

How much are 41 and 36?

How much are 46 take away 28?

How much money is on my desk? (Place any amount up to $1.00 on the desk.)

What time is it now? (May place a clock dial on the board.)

Draw a line that is about as long as your pencil. Draw another line right under the first that is only one-half as long as the first.

Sixth Grade

Write the number one million two hundred thousand thirty.

Subtract these:

$$\begin{array}{r} \frac{1}{2} \\ -\frac{1}{4} \\ \hline \end{array} \qquad \begin{array}{r} 3 \\ -2\frac{1}{3} \\ \hline \end{array} \qquad \begin{array}{r} 3\frac{1}{2} \\ -1\frac{3}{4} \\ \hline \end{array}$$

Prove by use of a diagram that $\frac{1}{2}$ and $\frac{3}{4}$ are equal to $1\frac{1}{4}$.

Round these numbers to the nearest hundred. 297, 508, $116\frac{7}{8}$, 241.9.

About how high is the door knob from the floor?

What street or building is about one mile from this school building?

Attention is called to the absence of basic addition or subtraction facts in example form in the first-grade test. Where such facts are used, they appear in problems. The understanding of quantitative situations of the problem type and understanding of common measures are especially important in teaching procedures where problems are used to illustrate new facts and procedures.

In the upper elementary grades these inventory tests furnish an introduction to the content of the arithmetic review. Especially good learning exercises can grow out of the misconceptions that the children exhibit on items that require the use of common measures. If these inventory tests are not to hamper teacher initiative and thereby make the review period a formal set of exercises, teachers must know what has been taught in preceding grades and must be left free to eliminate any items that are not considered in line with what the children will need in the coming year's work.

Standardized Arithmetic Tests

For a good many years a number of standardized arithmetic tests have been on the market. For the most part, these tests consist primarily of a section made up of examples dealing with the fundamental processes and a section of problems — or, as it is sometimes called, a reasoning section. Minor parts of the test may refer to recognition of geometrical figures and numerical terms. Standardized tests of this type have served and do serve some useful purposes. It should be recognized, however, that they have done some harm.

The greatest shortcoming of these tests lies in the fact that they have emphasized or fostered an emphasis on one kind of arithmetic (exact computation) to an extent far out of proportion to its real place in life. In fairness to the tests, it should be stated that teaching programs outlined in texts place just as heavy emphasis on exact pencil-and-paper computation as do the tests. We have already noted in the discussion on textbook testing programs (page 329) that tests often make for poor work habits because of the procedures that children must follow if they are to make the highest possible mark on the test. Reference is here made to the fact that the pupil taking a test is actually penalized for checking, carefully studying difficult situations, and employing other commendable but time-consuming procedures. Then, too, as is true of standardized tests in other fields, much harm has resulted from the misinterpretation of the meaning of scores of standardized arithmetic tests, both by teachers and by administrators.

A brief description of a common procedure following a test illustrates the point. Test scores were low; a quick survey of the errors of the children was made. It was noticed that addition, subtraction, multiplication, or division examples were missed. Accordingly, an extensive drill period dealing with these areas was prescribed as a remedial procedure. While such a drill program may produce slightly better results when

the next standardized text is given, the results will still be disappointing. Drill cannot make up for poor initial instruction. Instead of giving the children an opportunity to have experiences that will lead to the understanding that should precede drill, the drill procedures often produce still more confusion or misunderstanding of what arithmetic really is.

Mention has already been made of the fact that standardized arithmetic tests unduly emphasize exact pencil-and-paper computation. Furthermore, the computation emphasized occurs primarily in examples and not in the more important problems or reasoning section of the test It should also be noted that the actual computation required in the reasoning section of tests is relatively much simpler than that needed in the examples section. This condition has resulted from the fact that the test-makers, knowing that children are very weak in problem-solving, have set comparatively easy problems in order to keep the problems section of the test from being a mere waste of paper. Such a state of affairs is in itself pretty good evidence that our present arithmetic programs are not functioning properly. As far as tests are concerned, little can be done about the children's inability to handle problems. Tests can place a much greater emphasis on problems, however, by increasing the relative amount of time given to them. The examples section of tests might well be drastically reduced, since it is presently overemphasized and misinterpreted in a way that has led to bad teaching practices.

In order that tests may more nearly reflect the total arithmetic program, items that deal with aspects of the program other than computation in examples and problems must also be included. If the arithmetic program is a meaningful one, good tests will be concerned, not only with computation, but with the nature of processes, the significant aspects of the number system, the making of judgments, the understanding of problems (not just through the getting of answers), the use of

unwritten arithmetic, and so on. Examples of items that attempt to measure understanding in some of these areas are given below.

1. In the multiplication example

$$\begin{array}{r} 43 \\ \underline{24} \\ 172 \\ \underline{86} \end{array}$$

why is the 6 placed under the 7?

(a) Because arithmetic books say so.
(b) Because it's the second time you multiply.
(c) Because 2 tens times 3 is 6 tens.
(d) Because 2×3 is 6.

2. In the word *fifty-three* what does the "ty" mean?

(a) Tens.
(b) Has no meaning, but is used to make the word rhythmical.
(c) Means to add 50 and 3.
(d) Means to combine 5 tens and 3.

3. In which of these is the figure in the hundreds place underlined?

(a) 1$\underline{5}$,284 (b) 1$\underline{9}$26 (c) 1$\underline{5}$6 (d) 10$\underline{0}$

4. In adding the numbers $\begin{array}{r} 27 \\ \underline{38} \end{array}$ we say "8 and 7 are 15; put down the 5 and carry 1." Why do we carry 1 instead of 10 when 15 is equal to 5 and 10?

(a) Because 1 is easier to add to the next column.
(b) Because 1 in the next column means 10.
(c) Because it really is only 1 in the 15.
(d) Because that gets the answer.

5. A 1200-lb steer is about equal in weight to how many men?

(a) 3 (b) 25 (c) 8 (d) 12

6. In an election, candidate A received 73,281 votes, while candidate B received only 35,819 votes. Which would be the best way of describing how badly B was beaten?

(a) A won by at least 30,000 votes.
(b) B was thousands of votes behind A.
(c) A received a majority of the votes.
(d) A beat B "two to one."

For additional examples of items of this type consult Test D of the Basic Skills Battery.[1]

The fact that test items of this kind are almost unknown in standardized arithmetic tests is an indication of the difficulty that the test-maker encounters when trying to measure in areas of arithmetic other than exact computation. The easily constructed type of test item is always found in abundance, but rarely a large number of those that are difficult to construct. Test-makers who are willing to construct items that deal with understanding and the more significant aspects of number have been hampered by the fact that pupils have been so poorly taught that they offer nothing to measure. The test items thus fail to discriminate among pupils. For example, item 4 of the list of illustrative examples was answered correctly by only 16 per cent of sixth-, seventh-, and eighth-grade pupils. Thirty per cent chose response c while 18 per cent chose response *d*. Item 1 was answered correctly by only 18 per cent of sixth-, seventh-, and eighth-grade pupils. In interpreting these figures, it should be remembered that the per cents were not corrected for guessing.

For a more extensive discussion of the problem of evaluation in the field of arithmetic the reader should consult Brownell's chapter on evaluation in the *Sixteenth Yearbook* of the National Council of Teachers of Mathematics. Along with other aspects of measurement Brownell offers some commendable testing

[1] *Iowa Every-Pupil Tests of Basic Skills: Test D, Arithmetic* (Boston: Houghton Mifflin Company, 1940, 1941, 1942, 1943).

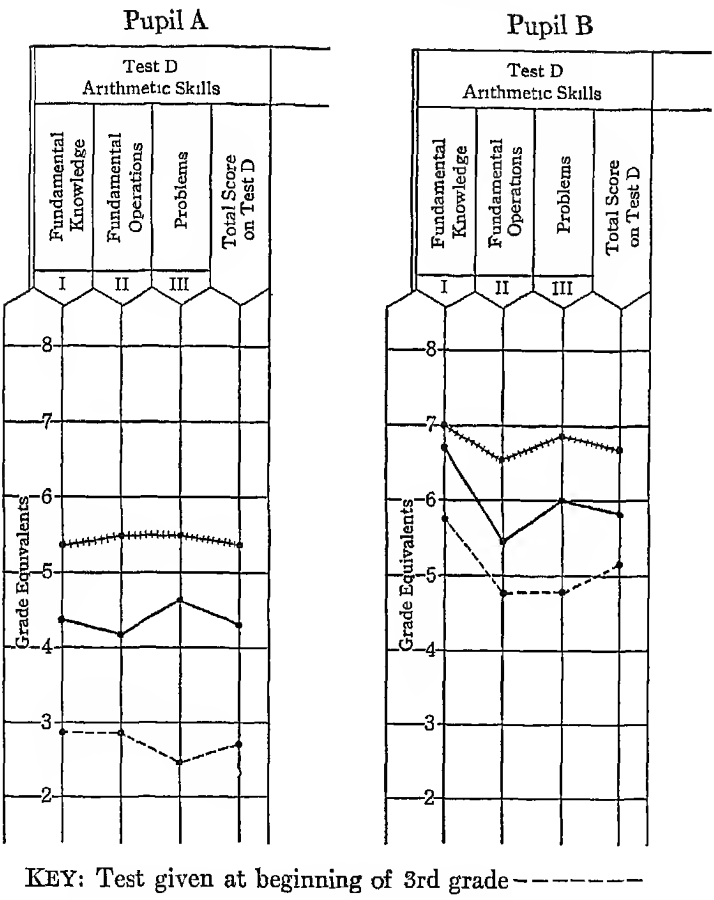
devices. Much refinement of some of these testing procedures is needed, however, before they can be used extensively in standardized tests. Regardless of the difficulties encountered in the construction of tests and of the poor items that may result from attempts to measure understanding in arithmetic, work in this field should be encouraged in every way possible. Unless standardized tests can be built to include items which measure understanding, such tests must be relegated to a very minor place in modern instructional procedures.

Now that we have discussed the shortcomings of the content of standardized tests, the harm that has been brought about by these tests, and the way to improve them, we should consider the general place of standardized tests in the total instructional program. In all educational work it is desirable to have data that can be used in evaluating teaching procedures. The results of a good standardized arithmetic test are an important part of the data that teachers and administrators can use in evaluating their program. Quite obviously, any data used in the evaluation of teaching must be reliable if the interpretations based on them are to be sound. Since all measurement is a matter of comparison, relative and not absolute, the reliability of norms and scores becomes doubly important. It is a fact well known to students of testing that achievement of a certain grade in a single school varies from year to year. Consider, then, the misinterpretation that might result if a pupil's score were compared with the performance of pupils in a markedly inferior class. Unquestionably the norms of a standardized test are more reliable than the subjective standards that the teacher sets or than the norms developed in the local school system. Standardized tests are the only means available for plotting on a chart the growth of the child in arithmetic from year to year. These charts, sometimes called profile charts, are especially valuable for giving the child an idea of the progress he is making. Although only a few standardized tests on the

market at the present time incorporate this feature as a part of the test, it makes for one of the major benefits that may be derived from a standardized test. Naturally, if the profiles of year-to-year growth are to be reliable, the tests must be reliable measures. Frank discussion with the child of the profile of his year-to-year growth is one of the best ways of developing a proper attitude toward study. After all, it is the child who must do the learning and who must recognize his weakness and his strength. A very commendable feature of the pupil-growth chart is the fact that the poor pupil's chart is as likely to show evidence of growth as is the chart of the good pupil. Often the amount of growth made in one year by a poor pupil is greater than that made by a good pupil in the same year. Since the poor pupil knows so little at the beginning of a year period as compared with the good pupil, the poor one's chances for growing are relatively better. For example, pupil A (see illustrative chart on page 339), whose rating was only 2.6, encountered many things to be learned during his study of third-grade arithmetic, while pupil B, whose rating was already 5.1 at the beginning of the year, probably encountered a relatively small number of things to be learned in the third grade.

Standardized tests can be used as the first step in a program of remedial instruction. Through a discussion of the test results, the need for more careful work in certain areas can be shown. Of course, the work that is to follow should not be all drill. The first remedial procedures should aim for the development of understanding. In the lower grades this aim will require such tests of understanding as the illustration of the situation through diagrams and the proof of a fact or process through object manipulation or through counting. The standardized tests referred to in this chapter are of the survey type and therefore cannot be considered sufficiently diagnostic to locate a particular fact or skill where a pupil's knowledge breaks down. As a matter of fact, there are diagnostic tests designed to give such informa-

Sample Profile Charts



KEY: Test given at beginning of 3rd grade - - - - - - -
Test given at beginning of 4th grade ————
Test given at beginning of 5th grade ++++++++++

From profile charts for *Iowa Every-Pupil Tests of Basic Skills*, on file in the University Elementary School, Iowa City.

tion. The time and effort required to give and score such tests are often out of proportion, however, to the value gained from the information that the test results provide. Only in a few exceptional cases are such tests of value. For further and more detailed discussion of the uses of standardized tests, books on educational measurement should be consulted.

A question frequently asked is, How often shall standardized tests be given? Certainly not more than once every three years if the only need fulfilled is that of satisfying someone's curiosity. However, if test results are correctly used, one standardized test a year is very desirable. Where some experiment is under way, two tests a year are essential.

In the selection of standardized tests, content should be the first factor considered. In addition to the content of the tests, attention should be given to the objectivity of scoring, the time and effort required of the pupil in taking the test, and the ease and time of scoring the tests. Since the measurement of some phases of arithmetic is more direct than that of many subjects, the early introduction of objective tests had very little effect on testing in arithmetic. The rapid progress of time-saving and effort-saving scoring devices in testing in other fields has resulted, however, in attempts to simplify the scoring of arithmetic tests. The Co-operative Test in Arithmetic has for several years been of the multiple-choice type. Extensive studies have shown that such tests, properly constructed, are reliable measures of arithmetical achievement. The 1942 eighth-grade examination distributed by the Iowa State Department of Public Instruction, a test of this type, varies from the Co-operative Test in that one of the four responses is not numerical, namely, "correct answer not given." If a separate answer sheet and a stencil-type scoring key are used in connection with the multiple-choice type test, a great saving of time and energy in scoring the tests can be effected. Any such saving should be welcomed, since it leaves the teacher free to do something about

the conditions revealed by the tests. Teachers are often so exhausted from scoring a test battery that they have little energy or desire to change their teaching methods in order to take advantage of the information provided by the test results.

The directions for a multiple-choice test, two sample items, a sample answer sheet, and a stencil-type key are illustrated below.

Directions: For every item in this test there are three possible answers and an N. The N means that none of the answers is correct. First do the solution required in the item and then select from the three answers the one that agrees with your solution. If none of the three answers is the same as yours, N should be chosen.

Mark your answers on the answer sheet in the following way: If you think the first answer to an item is the correct one, place an X in the first parenthesis; if you think the second is the correct answer, place an X in the second parenthesis; and so on for the third answer and the N.

Sample Items

1. Bill bought a tablet for 10 cents and two 8-cent pencils. He gave the clerk a half dollar. How much change should Bill get back? (*1*) 18 cents. (*2*) 26 cents. (*3*) 32 cents. (*4*) N.

Sample Answer Sheet	*Stencil Type Key*				
1. ( ) ( ) ( ) ( )	1				○
2. ( ) ( ) ( ) ( )	2			○	
3. ( ) ( ) ( ) ( )	3.			○	
4. ( ) ( ) ( ) ( )	4.		○		
5 ( ) ( ) ( ) ( )	5.	○			
6. etc.	6.			○	

The circles on the stencil key represent punched-out holes. When such a key is correctly placed on the answer sheet, these holes will be over the parentheses corresponding to the correct answers. Then, to find the total number of correct responses the scorer has only to count the X's that appear.

STUDY QUESTIONS

1. The usual teacher-made test item requires the child to find the one right answer. In what way is that a limitation? (*1*) It isn't. (*2*) It makes for too high objectivity. (*3*) It places undue emphasis on exact computation (*4*) N.

2. The usual teacher-made test is concerned with only one phase of arithmetic. Why is this undesirable? (*1*) After the first item is solved, solution of the others is not a test. (*2*) It is not representative of what the children have just been doing in arithmetic. (*3*) Such tests are diagnostic and only on an individual basis are they effective (*4*) N.

3. Why are the scores made by pupils on the usual teacher-made test spuriously high? (*1*) Because the children are familiar with the teacher's way of phrasing items. (*2*) Because the pupils have only to remember the steps in the recently taught computation. (*3*) Because teachers tend to construct items that deal only with those phases of arithmetic that they know the children have mastered. (*4*) N.

4 Why should arithmetic tests include items which require the use of approximate relationships? (*1*) In order to acquaint the pupil with that phase of arithmetic (*2*) In order that the test may represent what has been taught. (*3*) Such items are an excellent test of the pupil's ability to compute. (*4*) N.

5. Is it advisable to permit children to see and to discuss the test scores of the class members? (*1*) Yes (*2*) No

6. Why is the time given to testing in arithmetic frequently excessive? (*1*) Because it is so easy to diagnose the pupils' weaknesses by studying test results. (*2*) Because the tests are so easily scored and therefore require little of the teacher's time. (*3*) Because the pupils can take the tests in a very short time. (*4*) N.

7. What is the most serious criticism that can be made of the usual textbook testing program? (*1*) The norms are too high. (*2*) The different tests are not varied enough in content to give adequate measures of a pupil's whole achievement. (*3*) Almost all of the items require exact computation. (*4*) N.

8. What is the most serious criticism that can be made of most standardized arithmetic tests? (*1*) Practically all the items require exact computation. (*2*) The time required for giving them is too long. (*3*) Too much emphasis is put on the reasoning section. (*4*) N.

9. Why is the builder of a standardized test using poor judgment if he includes some items that are too difficult for 99 per cent of the pupils? (*1*) Because the test will discourage most of the pupils. (*2*) Because these items will have no part in ranking pupils. (*3*) Because items that are difficult are either tricky or require lots of time (*4*) N.

10. Teachers frequently are disappointed when the average score on a standardized test made by their pupils is only 60 per cent of the possible score. Is that feeling of disappointment a justifiable one? (*1*) Yes. (*2*) No.

11 If more of the superior students fail on a test item than do inferior students, should that item be retained in a test? (*1*) Yes. (*2*) No.

12. What is the most important outcome to be derived from the use of a standardized test in arithmetic? (*1*) It enables the teacher to diagnose the shortcomings of the pupils. (*2*) It provides the evidence on which to build a remedial program. (*3*) It provides a means of comparing the class with other classes (*4*) N.

13. What feature of the recognition type of arithmetic test makes that type of test highly desirable? (*1*) The scoring is objective. (*2*) The ease of scoring. (*3*) The thinking of the pupils can be controlled. (*4*) N.

14. For which type of student is the pupil profile chart especially useful? (*1*) The high ranking. (*2*) The low ranking. (*3*) The pupil who ranks about average. (*4*) N.

15. How often should standardized tests in arithmetic be given? (*1*) Every other year. (*2*) Every year. (*3*) Twice every year. (*4*) N.

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2. Greene, H. A., A. N. Jorgenson, and J. R Gerberich. *Measurement and Evaluation in the Elementary School.* New York: Longmans, Green and Company, 1942.

3. *Manual for Interpretation of Iowa Every-Pupil Tests of Basic Skills.* Boston: Houghton Mifflin Company. Form N.

4. National Society for Study of Education. "The Measurement of Understanding," *Forty-fifth Yearbook*, Part I. Chicago: University of Chicago Press, 1946.

5. Wilson, G. M. "Choosing and Using Standardized Tests in Arithmetic," *Education, 60*: 177–79 (November, 1939).

13

History of Number and Recreational Arithmetic

The Case for Teaching the History of Number

Supplying knowledge of the development of number and of measures as a basis for better understanding of our civilization was listed in Chapter 2 as one of the four major contributions that arithmetic has to offer to the education of youth. Because so few people know anything about the history of number, the claim made for the value of this phase of arithmetic must be considered highly theoretical. Therefore, the possibilities of teaching this field ought to be carefully examined before an attempt is made to incorporate it in the instructional program.

Since numbers play such an important role in modern society, it seems logical to assume that knowledge of the development of numbers should make for a better understanding of modern society. It is also a well-known fact that the people who are competent in a given area possess much historical knowledge of that area. Stories of number development interest and fascinate both children and adults almost as much as do stories of political struggles and should therefore merit some consideration. Furthermore, it is quite likely that a better understanding of modern uses of numbers will result from a knowledge of

how the ancients used them. Numbers and number work are probably appreciated more by those who know something of the history of number than by those who have no such knowledge.

Other values can be claimed for a study of the history of number, and certainly each of the given reasons could be much expanded. However, an extensive discussion would prove most convincing to the student who is already familiar with the subject. For the background essential to consideration of the history of number, the reader is referred to such books as those by Sanford, Smith, Smith and Ginsburg, Conant, Dantzig, Wheat, and Karpinski.[1]

What to Teach and How to Teach

A quick survey of these brief histories of number will reveal much material that will not be of value to children, although it might be very interesting to teachers. It is necessary, therefore, to select the phases of number that are to receive the attention of the elementary-school child. For the sake of brevity and to permit a better presentation, the following areas of arithmetic are proposed as being of historical significance: (1) counting; (2) numerals and numeration; (3) notation; (4) the development of zero; (5) old ways of reckoning; (6) number mysteries, including magic squares; and (7) the development of weights and measures. Each of these areas needs further definition, and the list should be considered only a tentative one.

Counting includes finger and object counting as practiced by various primitive peoples, the use of a base, as in the setting aside of one man or a large stone to represent a ten. In addition to base 10, other bases will be introduced through the study of certain primitive peoples of today. Naturally, the use of a base comes late in the history of counting. This area should include also the abacus and other counting devices. The

[1] See page 305, Selected References.

abacus will be especially interesting and useful in presenting the story of the development of other phases of arithmetic. Examples of such use are shown in teaching the discovery of zero and in demonstrating ancient ways of calculation. The Apache Indians' method of counting, involving use of a bag of stones, is representative of the object method of counting. The tally method, too, should be included.

The field of numerals and numeration is rich in historical material. The various ways of recording quantities through the ages never fail to arouse interest. In one school, the bulletin board that attracted the most attention during the school year showed how the numerals were written at various intervals from 450 A.D. to the present time. Children never fail to exhibit interest in the Egyptian method of writing quantities, especially their symbol for 1,000,000.[1]

The many variations in the writing of numerals indicate the rather limited use of numbers in the past. Moreover, the larger the numbers, the greater were the variations. Since only a few people used the large numbers and only scholars understood them, there was little incentive to establish uniformity. This area of numerals and numeration includes the ancient Hindu, Greek, and Anglo-Saxon methods of reading numbers as well as the various methods of writing decimals.

All Americans are familiar with the Hindu-Arabic and the Roman methods of notation, but few are familiar with other systems or know the major characteristics of various systems. Even a small knowledge of other methods of notation enables one to appreciate more fully the simplicity of our own scheme.

The development of zero might properly be classified under the heading of both notation and numerals. It is so important, however, that it is assigned a separate place in the tentative list of significant historical areas to be taught. The illustrative

[1] Louis C. Karpinski, *The History of Arithmetic* (Chicago: Rand McNally and Company, 1925), pp. 1–2.

lesson (page 350) gives the main points in its origin. Attention is called to the fact that the idea of a place-holder (zero) is much older than is indicated by Wheat[1] and Smith[2] — both of whom place the date of the discovery of zero after the first century A.D. Chiera[3] finds clear indication that the Babylonians had and used the idea of a place-holder or zero. Karpinski[4] credits the Babylonians with having knowledge of the principle of place value. The idea of zero was also developed by peoples of Central America.[5]

Under old ways of reckoning would be included the old Hindu methods of adding, use of the abacus, the sieve method of multiplying, a little duplication and mediation, and a few other methods.[6]

Number mysteries and pleasantries are so extensive that to make only a partial list might be misleading. Certainly lucky and unlucky numbers are of such universal interest that they should receive some attention. For other possible topics in this field, books by Dantzig[7] and Smith and Ginsburg[8] should be consulted.

Although there is an interesting story connected with the development of almost every one of our standard measures, only a few of them have been incorporated in arithmetic text-

[1] H. G. Wheat, *The Psychology and Teaching of Arithmetic* (Boston: D. C. Heath and Company, 1937), p. 75.

[2] D. E. Smith, *History of Mathematics* (Boston: Ginn and Company, 1925), II, 69.

[3] Edward Chiera, *They Wrote on Clay* (Chicago: University of Chicago Press, 1938), p 155

[4] Louis C. Karpinski, *op. cit*, p. 8.

[5] Anne Terry White, *Lost Worlds* (New York. Random House, 1941).

[6] As a source of these methods, the reader is referred to D E Smith, *History of Mathematics*, vol II, and Florence Yeldham, *The Teaching of Arithmetic Through Four Hundred Years* (London. Harrap and Company, Ltd , 1936).

[7] Tobias Dantzig, *Number, the Language of Science* (New York The Macmillan Company, 1930).

[8] D E. Smith and Jekuthial Ginsburg, *Numbers and Numerals* (New York: Bureau of Publications, Teachers College, Columbia University, 1937).

books. Before trying to find new stories and adapt them for children, we should use properly those that we have. They are not extra facts of arithmetic to learn; they are stories to be enjoyed just like other stories. As will be pointed out in the section on how to teach (see pages 350 ff.), many activities will arise out of the consideration of the stories of measures. The measures of time warrant special attention.[1]

While the seven fields briefly identified in the preceding paragraphs undoubtedly represent some of the major historical material, no particular claim is made for this list. It has not been tried out in school systems, and it must still be considered theoretical. In order that teachers and supervisors may have an opportunity to try out more quickly the historical phase of arithmetic and thereby be sooner in a position to talk something besides theory, a few suggested teaching procedures are offered. All of these procedures have been tried in experimental schools.

As hinted earlier in this discussion (page 345), the historical phase of arithmetic is something to be used and studied in a manner quite different from the approach to ordinary arithmetical material. Some of the stories may best serve their purpose when they are only read or told. As in the case of good factual stories in other fields, discussion will frequently be desired and enjoyed by the children. Perhaps the best use is made of a story when knowledge of the facts involved is required by some project, such as the preparation of a bulletin board display, the writing of an article, the dramatization of the story, or the portrayal of some phase for a notebook. The

[1] For references on measure, the following list will provide a good beginning: American Council on Education, Committee on Materials of Instruction, *The Story of Time* and *The Story of Weights and Measures* (Washington, D. C., 1932), National Youth Administration, *Standards*, The Modern World at Work, No 2 (Washington, D. C Superintendent of Documents); Edward Nicholson, *Men and Measures* (London: Smith, Elder and Company, 1912); D. E. Smith. *History of Mathematics*, II, chap IX.

lessons and projects that follow illustrate different ways of teaching historical phases of number.

Illustrative Lessons

Lesson 1 (Second Grade)

After reading to the class a description of the Apache method of counting, the teacher said, "I wonder if someone could count the children in the room, using the Apache method. Here are some stones you can use and you can use this box for the Apache's leather poueh."

Several children attempted to count by the method while the class watched. The story was re-read several times at the request of different children or as the teacher challenged the correctness of some step.

Lesson 2 (Third or Fourth Grade)

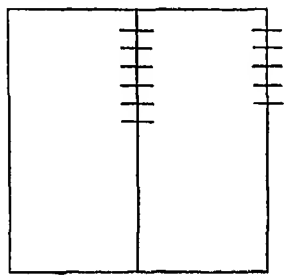
The development of zero. This lesson should come after lesson 18, Chapter 4.

"A few days ago you learned that a place-holder made the reading and writing of large numbers very much easier. You also learned that some people including the Romans did not use a zero or a place-holder. They did not use it because they did not know about it, and therefore did not know how much easier a plaee-holder would make work with numbers. The zero is so important that today some of you in the seeond grade ean write and use numbers better than many Roman men could do. Let's try adding some Roman numbers. Write these on your paper: twelve, eleven, fourteen, eighteen, and ten." The teaeher wrote the Roman equivalents on the board. After an attempt to show how addition could be performed with Roman numerals, the teacher let the children write the Hindu-Arabic numbers and then add.

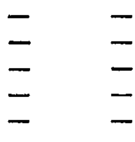
"Sinee the idea of a place-holder is so important, I think we will take time to learn how it might have been diseovered. The

Romans did not add the way you tried to do. They used the abacus. You know that the abacus does not leave a record. Some people of long ago would draw a picture of the abacus to show what amounts had been written on it. Suppose, for example, the amount 65, that we had a while ago, is to be recorded. Who can draw an abacus showing that?"

A child drew the following:



Several other amounts were recorded. The teacher then explained that some busy people left out the abacus lines. Several amounts were then recorded following this plan. The 65 then appeared thus



while the amount 103 appeared thus

The teacher wrote the record for 103 on the board without leaving much space in between the two sets of marks. Her record when compared with the children's and with other numbers recorded on the board made it quite clear that something was needed to show that the one mark was in the hundreds place. The children, of course, suggested our zero. The teacher

explained that various marks had been used, but that the situation did show how the zero might have been discovered.

Lesson 3 (Third Grade)

"All of you know that after you count the first 9 you count by 10's. Had you ever thought about why we count that way? Some people say that it would have been better if man had learned to count by 8's instead of 10's.

"The reason why we count by 10's is a very simple one. It's so simple that you will be able to understand it and even surprise your mother and father, because I doubt whether they know. If you already know, you won't be interested in the story in this book. Don't tell the others because they will enjoy finding the answer. I'll put the book here; when you have some free time you can read it."

The teacher then wrote on the board the question, "Why do we count by 10's?" [1]

Several days later the teacher again asked the question and then let the children tell their solution. In the discussion some actual finger counting was done. The fact that it was easier to think and see one or several pairs of hands than ten, twenty, or thirty fingers was frequently mentioned. As a summarizing step, it was suggested that some of the children might write a news story telling what they had learned and illustrate their description with appropriate drawings. The completed stories and illustrations were placed on the bulletin board.

Lesson 4 (Fourth Grade)

"When you were in the first and second grades you often made marks or used objects to show what numbers meant. Then in the third grade you sometimes drew pictures to show

[1] D. E. Smith, *The Wonderful Wonders of One-Two-Three* (New York: McFarland, Wade and McFarland), pp. 13–16.

quantities. Even now when you prove, you frequently draw pictures. The other day one of you drew 6 stick men each with half an apple to show that in 3 there are 6 halves. That picture of the stick men reminded me of the way the Egyptians wrote numbers. I thought you would like to see and hear about some of the Egyptian numbers. So instead of studying today, let's look at Egyptian numbers. I have a book here that shows how they were made. This is a 1. These are 2, 3, and 4. Do you see any resemblance to our numbers?"

Many numbers were presented in this fashion. When the teacher showed them the Egyptian symbol for 1,000,000, she said, "Why do you suppose they used a symbol like that? Suppose, as the last thing that we do with Egyptian numbers today, you write why you think the Egyptians used a drawing of a man with hands up in the air as the symbol for one million. Later you may see how nearly correct you are by looking on page 3 of this book."

Lesson 5 (Fourth Grade)

"On our frieze you have shown how farming has developed beginning at 4000 B.C. I have a book which shows how numbers were written about the year 5000 B.C. Do you think it would be interesting to present the development of numbers? If some of you would like to undertake such a project, here is a book which will give you the facts."

The project was undertaken by part of the class who drew the ancient numerals on pieces of $8\frac{1}{2}'' \times 11''$ paper. As a means of identification, the present-day numerals accompanied each sheet. Later one large sheet showing the numerals as written in the various periods was prepared. On several occasions the attention of the class was called to the project by means of some question or by discussion of some point of interest.

Lesson 6 (Fifth Grade)

A play,

How We Got the Name "Counter"

(*To be read by the announcer.*) Everybody knows how to count and most everyone would agree that a counter ought to mean a person who does counting. But the counter in a store is not even a person, much less a person who counts. We found in our study of number that there is a connection between counting and the counter in a store. We are presenting a play to show how the word *counter* became associated with stores and to show how the store counter and the person who counts are related

The first scene shows a store of ancient Baghdad That was, of course, before the days of mechanical adding machines or even before common people did any writing. Let's see how they figured the amount of the goods purchased in a store. To make it easy for you to follow the figuring, our money system is used.

(*Curtain opens with merchant sitting on the floor and a customer standing before him holding a number of things in his hand.*)

Customer: I have spent a pleasant three hours in your shop. These are the things I have decided to buy. Will you tell me how much I owe you?

Merchant: Most certainly, dear friend. We will put the amounts here on the counting board. Let's see. We agreed that this shawl was worth $2.40 (*Puts stones in the proper places on the counting board*) This wool, 20¢. (*Puts 2 stones on the 10's row.*)

Customer. No, I said that I would pay only 18¢.

Merchant: Very well, 18¢ it shall be. (*Picks up 1 tens stone and substitutes 8 ones.*) These ribbons you accepted at 83 coins and this belt at 94¢. Now let's see. We have more than 10 ones here. We will remove 10 and put 1 on the tens row. Now, take 10 of these away and put 1 on the hundreds row; now an-

other 10 away and another 1 on the hundreds row. You owe me $4.35.

Customer: That cannot be. I haven't purchased that much.

Merchant: Then we will count it again. (*Goes through the procedure of putting stones on the counting board, very slowly, explaining each step*)

Customer: Yes, I see you know how to use the counting board. Here is your money.

(*Curtain closes.*)

(*To be read by the announcer.*) Not all the merchants of Baghdad had counting boards, but all used the principle of the counting board Some merely drew lines on the ground and put the counting pebbles on the lines Still others used a board covered with dust. On this they drew lines and made cross-marks on the lines to show the number of 1's, 10's, and 100's The merchants of northern Europe did not like to sit on the floor, but did their counting on a table on which lines were marked. Our next scene shows an English store of about six hundred years ago. To make it easier for you to follow the figuring, the money system of the United States is used

(*Curtain opens showing merchant seated back of a table.*)

Customer: How much does this cost? (*Places an article on one edge of the table.*)

Merchant. That is $3.50 The other piece costs $1.85.

Customer: How much would I have to pay for 6 like this? (*Holds up an article*)

Merchant: Each one costs 35¢. (*Puts 3 on tens and 5 on ones; repeats for each, always moving up as 10 are reached until the six 35's have been recorded on the board.*)

Customer I'll buy all these things; how much do I owe you?

Merchant: Let's see $2.10 for the 6 (*puts on board*), $3.50 for this (*puts on board*), and $1.85 for this. That will be 1, 2, 3, 4, 5, 6, $7, and 1, 2, 3, 4, and 5 is 45¢. $7.45.

Customer. How much do these things cost?

Merchant: That is $.90, that is $1.10, and that $.65. All of that — let's see. Bring it over here. I can show you how I count it here. (*Merchant then records and adds the amounts on the counting board.*)

(*Curtain closes.*)

(*To be read by the announcer.*) The table eventually became the place where final examination of goods was made. Then as pencil-and-paper methods of addition developed, the counting table still had a useful place, for something was needed on which to do the figuring. The table was still called the counter. Gradually more and more tables were added to stores on which to show the goods that were for sale. Since all the tables were alike, they were all called counters.

. . . .

Space will not be taken here to describe possible lessons for the upper grades. The task of teaching the history of number is relatively much easier in the upper than in the lower grades, primarily because of the better reading ability of the older children, but also because of the greater maturity of their minds. The stories of the development of any of our standard measures or of the various ways of writing the decimal point are excellent topics to use at the upper-grade levels.

Summary

Every phase of the teaching of arithmetic is dependent on the teacher's grasp of the subject itself Unfortunately, there is no part of arithmetic about which he knows less than its history. The lack of adequate materials is partly responsible for this situation. The very nature of historical material, however, calls for a broader vision on the part of the teacher. A special plea is made to teachers, therefore, to do wide reading in the history of number. Unfortunately, as the histories of number have been written for scholars, they are not very practical books

for such practical people as the teachers of children. But for the present, such books will have to serve as the major source of materials.

In the field of historical arithmetic, there is an acute need for books for the teacher and for the child as well. When one considers that none of the books on the library tables of most elementary classrooms treats any phase of arithmetic, there is little wonder that that subject is listed as the drudgery subject. Historical number stories and books of number tricks and puzzles would help to make arithmetic more attractive to children.

Trick Problems of Our Grandfathers' Schools

Examination of arithmetic books of fifty years ago usually leaves the impression that arithmetic was very, very difficult. When this idea is coupled with the thorough personal dislike that so many teachers have for arithmetic, our grandfathers receive much unwarranted sympathy. While the arithmetic of 1890 was difficult and often as uninteresting as much of our arithmetic of today, there was a side to the arithmetic of 1890 that was challenging and entertaining. To those who mastered arithmetic, the long, puzzling, unreal problems of the textbook were a source of much pleasure and satisfaction. Students attacked the more difficult ones with zest and enthusiasm. To the problems of the book were added the definite puzzle, trick, and hard problems that were part of the stock and trade of every experienced teacher. The patrons of the school added to this fund of problem lore — especially when there was a young teacher to challenge. While these contributions, coming through the pupils as a "test of the teacher," were not always appreciated, they did add interest to the subject. Pupils spent hours trying to solve the most difficult problems or number tricks. Success in such an undertaking was as highly prized as first place in some athletic event or the leading part in a school play. Obviously, work with trick problems and puzzles would not

today be considered an efficient way to teach arithmetic. Even in our grandfathers' day it is improbable that much important arithmetic was learned through the solution of puzzle problems, but such problems were undeniably a means of motivation. Aside from the motivating factor, the puzzle type of arithmetic is good recreational material and merits consideration on that basis alone.

Universal Appeal of Puzzles

Interest in puzzles is in no way a characteristic peculiar to pupils of fifty years ago. Children of today are just as much interested in puzzles and number tricks. In fact, an otherwise dull arithmetic period can be made to sparkle by the use of a few good tricks or puzzles. The frequent use of puzzle situations by lecturers and writers and the continued popularity of crossword puzzles are good indications of the almost universal appeal of puzzles, for adults as well as children. Textbook writers have taken advantage of the general interest in solving the unknown type of problem. As Wilson has pointed out, many of the problems in modern textbooks are of the puzzle variety. This is, of course, a misuse, but the fact of its occurrence shows how broad is the appeal of tricks and puzzles. No better evidence of the interest in this kind of arithmetic can be produced than the long lists of such exercises that children bring to school once they recognize that problems of this type are acceptable. The teacher who sponsors such work often finds that through it home and school are bound more closely together. The children try trick problems on their parents, who in turn try out the children with other tricks.

Place for Recreational Arithmetic

The discussion above has emphasized the fact that number puzzles and number tricks have an appeal which makes them good recreational materials. It is to provide recreation and

not necessarily to teach arithmetic that the puzzle type of work is included in a number program. Note should be made of the fact that the words "not necessarily" are used. There are many people who sincerely believe that their insight into the possibilities of number and subsequent interest in arithmetic was brought about through the solution of some trick problem. For example, many adults who have worked the horseshoe problem (see problem 1, page 360) contend that no better way of teaching geometrical progression can be found. Children's interest in some of the concept-building exercises described in Chapter 4 is probably due to the puzzling nature of the problems.

Regardless of whether these claims have any value, there is something to be gained through the use of recreational materials. As has been pointed out, the arithmetic period becomes interesting when something challenging is introduced. This is well shown by the experiences of a teacher in a departmentalized system who was having difficulty in holding the interest of the children. The history, science, and geography teachers all had intriguing material to offer and were getting the best efforts of the children. This arithmetic teacher began to use number tricks and puzzle problems at the beginning and sometimes at the close of the period. Since proof in some form is an essential part of such work, the teacher had little trouble in shifting to the use of proof in regular arithmetic work. Before long many children were meeting the teacher before the beginning of the period and asking if they might present their number tricks. The arithmetic period was no longer the disliked or least liked period of the day. After the children's interest was caught, only about fifteen minutes a week were given to puzzles.

Unless special care is taken, an undue amount of time will be spent in puzzle-solving. To use more than thirty minutes a week, except in the introduction of such work, might lead to an arithmetic of the type of fifty years ago. Perhaps fifteen minutes a week, as used by the teacher referred to above, is

sufficient time to devote to puzzle arithmetic. However, the nature of such work precludes the assignment of a fixed amount of time per day.

The teacher using the method proposed in this book, where there is little demonstrating and telling, has a special need for recreational arithmetic. It restores to the teacher the honored role of "knowing more" that is so important to the learning situation. While the teacher may continually refrain from telling how or doing the work in the course of the regular teaching procedures, there is nothing to keep her from presenting and later solving for children the most difficult trick problems. Through such exercises, the teacher's prestige is enormously enhanced. Furthermore, the goodwill generated through the use of recreational arithmetic is of great value to the rest of the arithmetic period. Just as was true in the school of fifty years ago, pupils will make little distinction between recreational and "work" arithmetic. In fact, it would be a mistake to emphasize that the two are different types.

Types of Exercises

The exercises which make up the puzzle problem and number trick part of recreational arithmetic are not easily classified. A few of the exercises that have been successfully used are listed below, though no attempt is made at classification. The reader will find many more such exercises in the references.

1. A man rides into a blacksmith shop and asks that his horse be shod on all four feet. He wants to know how much the smith will charge. The smith quotes his price in the following manner: "I will charge one-fourth cent for setting the first nail, one-half cent for the next, and continue to double the amount for each nail thereafter." If eight nails to the shoe are required, how much does it cost the man to have his horse shod?

2. Three couples on their honeymoon came to a stream where there was only one boat with a capacity of two. The husbands were jealous and would not permit their wives to be

in the boat or on the ground in the presence of another man unless the husband himself were present. How did the couples cross the stream?

3. Write any number you wish. Multiply by 2, add 18, and then divide by 2 Now subtract the number you began with, and I'll tell you the answer.

Key: It is always 9

Sample: $23 \times 2 = 46$
$46 + 18 = 64$
$64 \div 2 = 32$
$32 - 23 = 9$

4. Choose a number and multiply it by 6, add 12, and divide by 2. Now subtract 6, and tell me your answer, and I'll tell you the number you started with.

Key: *To get the original number, divide the answer given by 3.*

Sample: The number thought of is 2.

$2 \times 6 = 12$
$12 + 12 = 24$
$24 \div 2 = 12$
$12 - 6 = 6$
(*Think*) $6 \div 3 = 2$

5. If you would like to see some rapid-fire addition, write down three nine-place numbers. Now, I'll write two more. See how quickly I can add them.

Key: In writing your first number, put down a figure that, when added to the corresponding figure of the first number written by the other person, will give a sum of 9. Follow the same procedure in writing your second number except that you use the second number written by the other person. To get the answer, write the third number with 2 subtracted from the last figure and 2 placed before the number.

$$\textit{Sample.}\quad \begin{array}{r} \left.\begin{array}{r} 445679311 \\ 234961192 \\ 114796852 \end{array}\right\} \text{numbers the other person writes} \\ 554320688 \\ \underline{765038807} \\ 2114796850 \end{array}$$

6. Write down your age in years (the age that you are now). Multiply this by 2 and then subtract 3. Now, multiply by 50 and add 39. Add the amount of change that you have in your pocket (less than one dollar). Tell me your total, and I will tell you how old you are and how much change you have in your pocket.

Key: To the answer given add 111 The first two figures will tell the person's age, and the last two will tell the amount of change in the person's pocket.

Sample: Take 26 as the person's age.

$$26 \times 2 = 52$$
$$52 - 3 = 49$$
$$49 \times 50 = 2450$$
$$2450 + 39 = 2489$$
$$2489 + 25 \text{ (change in pocket)} = 2514$$
$$(\textit{Think})\ 2514 + 111 = (26 \text{ years, and } 25 \text{ cents})$$

7. What has three feet and can't walk?

Key: A yardstick.

8. A bottle and a cork cost $1.10. The bottle cost one dollar more than the cork. How much did each cost?

Key: The bottle cost $1 05, and the cork cost 5 cents.

9 One morning a shoe merchant sold a pair of shoes for $5. In payment he received a $20 bill. He had to go next door to the grocery to get the change. The grocer gave him four $5 bills. Later the grocer discovered the $20 bill to be counterfeit, and asked the shoe merchant to make it good. He did. How much did the shoe merchant lose?

Key: $15 and the shoes.

10. With a 3-gallon measure and a 5-gallon measure, how could you measure out 4 gallons of water?

Key: First fill the 3-gallon measure, and empty it into the 5-gallon container. Then fill the 3-gallon measure again and pour into the 5-gallon can until it is full. There will be 1 gallon left in the 3-gallon can. Now empty the 5-gallon can and pour

the gallon from the 3-gallon can into the 5-gallon can. Then refill the 3-gallon container and empty it into the 5-gallon measure, and the 5-gallon measure will now contain 4 gallons, the amount asked for.

The compilation of materials to be used in a recreational program in arithmetic is an excellent project for an individual teacher or a group of teachers to undertake. Not only will they be collecting material of value to them, but through such a venture teachers will make pleasant contacts with parents and other friends of education. A good feeling toward arithmetic or school work is initiated when the teacher casually describes to the parent one of the room experiences and then asks if the parent remembers any number trick or puzzle that might be used. People, books, and even current magazines are sometimes a source of recreational items. The advertisement used by the Comptometer Company in the February 15, 1941, issue of *Collier's* is an excellent example of such material.

Thus far the recreational arithmetic program has referred only to the use of puzzles and tricks. There are other devices which will make arithmetic more enjoyable. Especially important is the history of number. Since that topic is treated fully in another section (pages 345–357) and since it is used for other purposes, it will not be emphasized in the recreational program. Attention is called, however, to the dramatization of phases of the history of number and to the writing of numbers in ancient fashion. The arithmetic match (similar to the spell-down in spelling), using the fundamental processes in example form, is always enjoyed by children who have gained some facility in the use of the processes. For an arithmetic match, two sides are chosen and the two captains begin the match by going to the board and working an example dictated by the teacher. The first to finish with the correct answer is the winner. The winner gets one tally or mark; the loser must sit down, and another from his side challenges the winner.

The games suggested for the primary grades are, of course, recreational in nature. However, it will be recalled that these games are included in the program because they aid in the building of concepts. It is doubtful whether the many devices used in teaching arithmetic have much value in the recreational program.

Magic squares and magic circles might also be included among the tricks and puzzles. Similar to magic squares are such number "pleasantrics" as the following:

$$1 \times 9 + 2 = 11$$
$$12 \times 9 + 3 = 111$$
$$123 \times 9 + 4 = 1111$$

and so forth.

For additional material that will prove valuable in connection with this phase of arithmetic, consult the references listed on page 365.

STUDY QUESTIONS

1. Why is the history of number not emphasized more in the teaching of arithmetic? (*1*) Because it is too difficult to teach. (*2*) Because elementary-school pupils are not interested in history. (*3*) Because it is not essential to efficient computation. (*4*) N.

2. For what reason are the arguments for the teaching of the history of numbers considered theoretical? (*1*) Because there are no exact answers to historical questions. (*2*) Because so little of this phase of arithmetic is actually taught. (3) Because the values to be derived from teaching are difficult to identify. (*4*) N.

3. In the Apache method of counting are number names essential? (*1*) Yes. (*2*) No.

4. If an abacus is used to do addition, is it necessary to know the basic facts? (*1*) Yes. (*2*) No.

5. If an abacus is used to keep records of quantities, is a place-holder needed? (*1*) Yes. (*2*) No.

6. Will a pupil's command of arithmetical operations be improved by learning to solve trick or puzzle problems? (*1*) Yes. (*2*) No.

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14

An Overview

The Program in Practice

The preceding chapters have presented in detail the various specific features of an arithmetic program. The description of each feature, such as the addition of whole numbers and the multiplication of tens, was presented as a unit. While such a unit presentation is essential, there is also a need for getting a brief picture of the total program with some of its various aspects woven together. In an attempt to present such a picture, this chapter offers a description of five successive lessons in several grades. In addition, an attempt is made to answer the questions most frequently asked concerning the program.

Five Successive Lessons for the Third Grade

The five lessons described here occurred at the sixth week of the school year. The work dealt primarily with the basic addition and subtraction facts. Each arithmetic period was approximately thirty minutes in length, and three meetings a week were scheduled.

Lesson 1

The first five minutes of the period were spent in an oral exercise which the teacher initiated in the following manner:

"While we are waiting for some members of the class, let's see if you can guess the numbers I'm thinking of. The person who guesses correctly may give the class numbers to guess. I'm thinking of two numbers that make 11."

Other situations presented to the class were two numbers equaling 9, 8, 15, 17, 12, 10, 20, 32, and 40. To some of the children who gave two numbers that did not equal the required number, the teacher said, "Will you take 8 sticks and 7 sticks and put them together and see if the total is 14?" To others the teacher merely said, "Prove your answer."

If the numbers above 18 were not guessed after four or five tries, the child was told to give the correct combination and then select someone to take his place.

At the close of the five-minute oral period, the teacher said. "For the remainder of your arithmetic period those of you who have unfinished work may do that. Here on the board you will find a list of the things you were doing last time. Those of you who are doing some proving, do that first. If you finished your work last time, bring it to me for a check, or let someone in the room check it first."

The following list was on the board· (1) Draw diagrams or pictures to show the addition facts of which you were not quite sure. (2) Study addition or subtraction facts from cards, practice sheets, or by writing them. (3) Find the answers to the problems on page 46 of your text. Then prove that your answers are correct. (4) Make story problems for the class to use.

After each of the four things to do were initials of the different children who were working on that task. Task 1 represented the lowest level of achievement of the list written on the board, and grew out of a self-test that the children had taken. Task 2 was for those students who had not yet learned the basic addition and subtraction facts so that they could respond to them automatically. The need for this type of knowledge had been

demonstrated to the children by a timed exercise that had been given earlier. Task 3 was, of course, a means of checking the understanding of the children. Those working on this task were for the most part advanced pupils, but there were some doing this work who were very much retarded. Task 4 was also primarily for the advanced pupils.

As the children worked on the different assignments, the teacher went about the room giving encouragement to some (e.g., "That's a good way to show that fact"), questioning others (e.g., to the child who had pictured 8 and 6 by placing marks for each quantity in one compact group, the teacher said: "Can you see that that is 8 very easily? I can't."), and on one occasion she discussed with several members of the group a particular method of proof that one child was using. When a child doing task 1 had finished, the teacher discussed all his illustrations with him. The discussion was directed toward answering the question, "Is that the best way to show that fact?" If the teacher thought the child had a satisfactory illustration or that he would not profit from further work of this type at the time, she suggested or made another assignment.

Lesson 2

An entirely teacher-directed oral period made up the first five or six minutes of this lesson. The things presented to the class were of these types: (1) How much is 18 take away 9, 8 take away 5, 14 less 6, and so on? (2) How many are 30 and 10 more, 20 and 2 and 6, 31 and 20, and so on? (3) In a pioneer story I read that two boys killed 8 squirrels one day, 21 the next day, 5 the next day, and 10 the next day. How many squirrels did they kill in all?

Since there was disagreement about the last exercise, the children were told that they might use pencil and paper and show just how they had thought in solving the problem. The first children to finish were told to put their work on the board,

to solve the problem using tens and ones blocks, to show the problem in another way, or to check someone else's work. In this way the slower workers were given additional time. When all had had a reasonable time to work the problem, the various solutions were discussed.

The teacher then said: "For your work today turn to page 48 in your text and do the practice exercises on that page. When you finish, here are some extra things to do." A few children were given special assignments like the following: (1) "Let Jane be your checker as you try to give the facts on these cards." (2) "You have studied the addition and subtraction facts quite a while now. Try this practice exercise." (3) "You have a good way of adding, but some of the people don't quite see how you do it. Will you make a drawing to show just how you make tens and ones out of two numbers?"

The extra assignments were as follows: "Copy and write the missing number. 24 = ... tens, ... ones; 36 = . . tens, .. ones; 4 tens and no ones = ...," and so on. Near the close of the period the teacher had a few children put examples of their work on the board. The last few minutes of the period were spent discussing these examples. The discussion was initiated and directed by such questions as: "Can you tell why Jim wrote 40 as equal to 4 tens and no ones?" "If there are no ones, why write anything?" "From the work that Mary has on the board, can you see how she thought?"

Lesson 3

"It is now time for arithmetic. Assignments are written on the board." (A procedure similar to that used for lesson 1 was followed.) "I want to help as many of you as I can, so let's not waste any time getting started."

The children worked on the assignments with the teacher, following the procedure described in lesson 1. As the work progressed, the list of tasks and the assignments of different

children changed. A new task was not written on the board immediately unless a large group of children were starting to work on that task. Toward the latter part of the period, the teacher had each of the children who had been making story problems dictate one of the problems to be written on the board. The last five minutes were spent in solving and proving these problems.

Lesson 4

At the beginning of the period, the teacher passed out cards on which the numerals from 1 through 9 were written. Two children were chosen to start the exercise. They came to the front of the room and held up their cards. The child who gave the correct sum was permitted to take the place of one of those at the front. Later, the exercise was varied so that three children and then four were displaying cards at one time.

After about four minutes had been spent with the card device, the teacher directed the attention of the class to four problems that she had previously written on the board. The object of using these new problems was to bring the group back together and to illustrate or emphasize some special point. The problems used were concerned with addition and subtraction facts. The children were told that they could work the problems without using pencil and paper; that they could use pencil and paper and use numbers; that they could use pencil and paper and draw; or that they could use objects. Those who chose the first two methods were asked to verify their answers. Some of those children (the more capable) who used the latter two methods were asked to use numbers to show how they had solved the problems. As the pupils worked on the problems, the teacher noted particular difficulties and suggested procedures like the following: (1) "You need to memorize some of the addition facts. In the next work period we have, you should

spend your time doing that." (2) "I see you know how to do subtraction, but you are certainly taking lots of time. Don't you think it would be better if you would practice on subtraction the next time we work?"

Lesson 5

"Yesterday in social studies we learned that we didn't know much about the size of an acre. John reports that an acre is a square about 208 feet on the side. I doubt whether many of us know how big that is. To help you get an idea of an acre, I've asked the principal if two of you might mark off an acre on the playground. Who would like to do it?"

After two children were selected, they were allowed a few minutes to tell the class how they were going to proceed. When class and teacher were satisfied, the two left the room with their measuring tape.

The teacher then said: "Instead of an oral period today, I want you to estimate some distances. Write five numbers in a vertical, up-and-down column on a piece of paper. As I give you the distance to measure you will write your estimate. (1) How long is your little finger? (2) How wide is this sheet of paper on which you are writing? (3) How wide is this room? (4) How high is this bookcase? (5) How high is the top of this chair from the floor? All of you can do your own measuring for the length of your little finger and the width of your piece of paper. I'll ask J. and M. to measure the room, R. and T. to measure the bookcase, and L. and S. to measure the chair. They will write their measurements here."

While the three pairs of children measured, the teacher and the class guessed at the length of objects that members of the class held up. After all measures had been made, the pupils used the remainder of the time for work on a set of practice examples in addition and subtraction.

The five lessons described above illustrate a number of important teaching procedures. Since these important procedures did not receive particular emphasis in the description of the lessons and because the principles underlying procedures are not always easily perceived, the most significant of them are listed and discussed briefly in the following paragraphs.

First, the learning procedures employed made extensive use of class spirit in motivating learning. Since there is much to be gained from such motivating factors, special measures to promote keeping the class a unit are important enough to be included in the teaching program. The oral exercises and group projects, such as identifying an acre of space, were important procedures used to maintain class unity. In spite of wide variations in achievement and ability, all members of the class took part in these activities. Then, in lesson 4, all pupils were given the same set of problems to work, yet individual differences in ability were cared for by permitting the pupils to work on the problems in different ways.

Second, an important procedure is illustrated by the problem assignment in lesson 4. In the preceding lessons, children had been working on individual or small group assignments. Often such assignments, although clearly understood by the pupils, begin to become rather tiresome and uninteresting. This new set of problems served as somewhat of a "reviver" of interest. The old tasks were discarded and new ones undertaken. It is true that even though many of these new assignments were almost the same as the old ones, they nevertheless gave the children the impression of starting afresh.

Third, a wide variety of content was dealt with through oral exercises. In this way, the children were given a sense of new things undertaken, although in reality they were just acquiring a better understanding and command of the processes of addition and subtraction.

Fourth, individual differences were cared for by the various tasks that the children were working on in the lessons. Although the major reason for some of the special assignments for fast workers was to gain time for the slower ones, these assignments also served as enrichment of experience. For example, the child who was asked to show by means of a drawing just how he added certainly had an opportunity to get a broader and more thorough view of what addition really is. In these lessons the idea of searching for other ways of showing a fact or a process was dominant. Out of such experiences children gain the concepts and understandings which will make for recognition of the best method. The use of objects and marks in demonstrating the truth of solutions is so much a part of the lessons that it needs no more than mention here.

This same type of work made up the major part of the work of this third-grade class until multiplication and division were introduced. Minor variations were furnished by consideration of some historical material and by special emphasis on a few standard measures. Other timed tests covering basic addition and subtraction facts, addition and subtraction with two- and three-digit numbers, and column addition were given in order to permit the children who had not mastered these phases of arithmetic to see the need for intensive study.

Five Successive Lessons for the Fourth Grade

These fourth-grade lessons followed immediately after the introductory lessons dealing with multiplication with two-place numbers described in Chapter 6.

Lesson 1

"Yesterday at the close of the period most of you thought you knew how to multiply with two-place numbers and that what you needed was practice in fixing what you had just learned. There are a number of examples which you can use

as practice on page 191 of your textbook. In case you have already forgotten how to multiply two-place numbers, I have put a problem on the board which you can use in figuring out the method." The problem was as follows: "Each of the bags of dog food weighed 20 pounds. What was the total weight of the 10 bags?" Following the problem were these directions: "If you do not see how to multiply to answer the question of the problem, add. Then write the numbers as tens and ones and try to figure out what you do when you multiply with ten."

After a period of work the teacher put the following on the board:

	2 tens	no ones
	× 1 ten	no ones
	no tens	no ones
2	no tens	

She then said: "All of you who are not sure about how to multiply with tens, look at what I have put on the board. Several of you had that written on your papers. What word should be written after the 2, tens or hundreds?" When there was some hesitation the teacher called attention to the fact that through addition the answer had been found to be two hundred. Therefore, the word "hundreds" should be written after the 2. In the discussion, the conclusion was reached that when you multiply a ten by a ten the product is hundreds or hundreds and thousands. To show that it is hundreds in the example you write two zeros.

The pupils then began to work on the examples, some of which contained tens and ones (e.g., 14) in the multiplicand. After working the examples, they were asked to prove that the products they had secured were correct. The only additional guide given to the pupils during this lesson was a notice, "When You Finish," written on the board by the teacher, containing this direction: "Write a statement which tells how the multiplication with tens differs from the multiplication with ones."

Lesson 2

"Today let's write answers to the oral arithmetic questions. Put 10 numbers on your paper. Remember we are not using pencil and paper to do any figuring, just to record answers."

1. How many are $8 + 8 + 5 + 10$?
2. What is $9000 + 7000$?
3. A man bought goods amounting to \$2.45. If he gave the clerk \$3.00, how much change should he receive?
4. How much are 8×40?
5. What is the sum of 600, 300, and 200?
6. What is the weight of a 6 ton truck in pounds?
7. How many inches in $2\frac{1}{2}$ feet?
8. What is $98 - 50$?
9. About how far from the floor in feet is the top of the bookshelf?
10. A farmer has 30 acres in oats, 60 acres in corn, and 10 acres in other crops How much crop land does he have?

Immediately following the reading of the last item, the teacher asked different members of the class for their answer to item 2. A little time was taken in letting some of the children who obtained an acceptable answer tell how they thought. A few of the other items were checked in the same manner, but the methods of thinking in the solution of all items were not requested. When all items had been checked, the teacher asked those who had made perfect scores to raise their hands.

The teacher then said: "Yesterday you did multiplying with tens. Here is another problem for which tens are multiplied. First, answer the question in the problem, then do the examples. Remember that if you cannot multiply you can add. Then try to figure out how you can get the answer by multiplying. Separating the numbers into tens and ones may give you an idea."

The problem required the multiplication of 23 by 12. The examples, too, included tens and ones in both multiplicand and multiplier. After some time had been allowed for solving the

problem, three pupils were asked to put their work on the board. With these examples as a basis for discussion, the class then tried to find the best way to multiply. Since addition had shown that the answer was 276, the task before the group was, "How can we multiply 23 by 12 to get 276?"

$$(a)\qquad \begin{array}{rll} & \text{2 tens} & \text{3 ones}\\ & \text{1 ten} & \text{2 ones}\\ \cline{2-3} & \text{4 tens} & \text{6 ones}\\ \text{2 hundreds} & \text{3 tens} & \\ \hline \text{2 hundreds} & \text{7 tens} & \text{6 ones}\end{array}$$

$$(b)\quad \begin{array}{r} 23\\ \times\ 12\\ \hline 276\end{array} \qquad (c)\quad \begin{array}{r} 23\\ \times\ 10\\ \hline 230\end{array} \quad \begin{array}{r} 23\\ \times\ 2\\ \hline 46\end{array}$$

$$230 + 46 = 276$$

Solution *b* was explained as follows: "$12 \times 3 = 36$. Write the 6 and carry the 3. $12 \times 2 = 24$. $24 + 3 = 27$." In commenting on this solution the teacher remarked that the method was not the one most people used and that it was rather difficult if the numbers you were using were large; for example, 18×8.

The teacher pointed out that both solutions *a* and *c* were longer than the best method that most people use, but could be easily changed to the best method. She then showed how the multiplication is done, and emphasized the relation between the long solution (*a*) and the best solution by such questions as, "How do you know that this 3 should be written as tens?"

Lesson 3

The first of the period was used for an oral exercise. This time, however, the children did not record answers. Several of the items in this oral exercise involved multiplication of even tens by ones and of tens numbers by 10 and 20. Following the oral exercise the following assignment was made: "Now

let's review what we did yesterday on multiplication with two-figure numbers. Turn to page 194 of your text. First, read the problem and the solution. See if you can figure out every step in the solution." The problem on page 194 of the text required the multiplication of 32 by 12. The solution was explained in detail, and the terms *partial products* used and defined. As soon as the pupils had had time to consider the text problem and solution, the teacher asked, "Does the solution of the textbook differ from the one we worked?" Only minor differences were noted. The teacher then said: "Then we have an explanation of multiplication with two-figure numbers in our text to which you can refer if necessary. The next thing you need to do is to practice multiplying two-figure numbers. Do the examples on page 194 and if you have time start those on page 195. If you have trouble, first try to figure out what to do by looking at the solution in the book. If you still can't figure out what to do, raise your hand and I will try to help you."

For the remainder of the period the pupils worked on practice examples.

Lesson 4

The teacher began this lesson by calling the attention of the class to the following directions written on the blackboard:

1. Solve this example

$$\begin{array}{r} 34 \\ \underline{\times\ 16} \end{array}$$

2. Why should the 4 in your second partial product be written under the zero?
3. Do $6 \times 34 + 10 \times 34 = 16 \times 34$?
4. Solve examples 1, 2, and 3 on page 196 of your text and prove that your work is correct.
5. Write the steps you use in multiplying 27 by 17.

In the discussion following a period of study, special attention was given to the ways of proving used and to the writing of the steps for multiplying 27 by 17. The two ways of proving readily accepted by the class were addition and a combination of multiplication and addition. The steps as developed in the class discussion were as follows:

Step 1

$$\begin{array}{r} 27 \\ \underline{17} \\ 189 \end{array}$$

Multiply 27 by 7. Write the ones (the 9) under the 7, the 8 in the tens place, and the 1 in the hundreds place.

Step 2

$$\begin{array}{r} 27 \\ \underline{17} \\ 189 \\ \underline{27} \end{array}$$

Multiply the 27 by 1 (1 ten). Write the tens (the 7) in the tens place and the 2 in the hundreds place.

Step 3

$$\begin{array}{r} 27 \\ \underline{17} \\ 189 \\ \underline{27} \\ 459 \end{array}$$

Add the two partial products.

Lesson 5

At the beginning of the lesson the pupils were directed to their textbook by the following directions given orally: "On page 198 of your textbook you will find an explanation of a good way to check multiplication with two-figure numbers. It's shorter than proving. See if you can figure out what is done."

In the class discussion that followed, the checking procedure (interchange of multiplier and multiplicand) was fully explained. Examples were then solved and checked by this procedure. Near the close of the period fifteen minutes were devoted to

review exercise which included multiplication with two-figure numbers, addition, subtraction, and division. The last three or four minutes of the period were given to an oral exercise. The items used in the exercise were similar to those used in lesson 2.

The next few lessons followed to some extent the pattern already described in the five preceding lessons. Differences in procedure were introduced through a lesson on the sieve method [1] of multiplication and the fact that both multipliers of ones and tens were used in some of the practice exercises. In the lessons of the next few weeks, the children were also given teacher-made tests which involved not only the procedures just studied, but also some work in addition, subtraction, and division.

In the lessons just described, the attention of the reader is called again to the use of oral exercises as a means of holding the class together, of pointing out significant or common number usages (see lesson 2, items 3 and 6), and of introducing and keeping alive procedures already studied. Another point of special note is the large amount of time given to initial learning procedures before practice for mastery or fixation is undertaken. While a great deal of time is given to practice, attention is called to the fact that the pupils were frequently directed to use procedures to test their understanding of the principle on which they were practicing. The testing procedures most frequently used were proving and demonstration of other ways of working the example. Attention is also called to the manner in which the children were directed toward the discovery of a new process. This example should dispel all belief in the notion that children must make the discovery without help.

[1] D. E. Smith and Jekuthial Ginsburg, *Numbers and Numerals* (New York: Bureau of Publications, Teachers College, Columbia University, 1937), p. 30, H. G. Wheat, *The Psychology and Teaching of Arithmetic* (Boston: D. C. Heath and Company, 1937), p. 108.

Questions Most Frequently Asked

Teachers and supervisors who have observed the method of teaching arithmetic described in this book often ask questions concerning the method. Seven of the most frequently asked questions with answers are given in this section.

1. *In a program where so much emphasis is given to developing understanding, do you need any drill?*

Related questions are: How much drill do you have? Do you believe in drill? Is there a place for drill? What shall we do about drill?

While the preceding lessons have shown very clearly the point of view of the author with regard to drill, the issues regarding drill are so frequently encountered that it seems worth while to discuss the matter further. There is a place for drill or practice in the learning of arithmetic. Indeed, to secure the best results, drill must be included in the program. After an important fact or process is understood and the child sees a need for knowing this fact or process, no better teaching procedure can be followed than permitting the learner to practice the newly acquired skill. Such practice fixes the procedure, thereby eliminating the necessity for the careful, painstaking thought which requires energy that could be better directed toward the solution of the number situation confronting the children. As was stated in the discussion of intensive study of the basic addition and subtraction facts, automatic mastery of important facts and processes is essential and drill or practice is one of the most economical ways of acquiring mastery.

There are, however, a number of principles governing the use of drill which should be carefully observed in order to make it an asset rather than a handicap in the teaching of arithmetic. First, and most important, drill or practice on a fact or process should come after understanding has been acquired. In the

average classroom not all children will acquire understanding at the same time. It will be necessary, therefore, either to start the use of drill procedures before all have acquired understanding or to have those students who have already acquired understanding mark time. While either of the two alternative procedures is far from desirable, the former is probably the one to use. During drill frequent attempts should be made to check understanding. This can be done by asking children to demonstrate the truth of the examples they are solving. Since not all children learn at the same rate, there should be relatively few occasions when all pupils in a given class practice the same thing for any extended period of time. Furthermore, if drill is not to come until understanding has been attained, a relatively long time should elapse between the initial introduction of a process and the time of the practice period. For that reason the explanation of more than one illustrative problem will usually be needed.

A second important principle guiding drill is that children must see a need for the drill. In the learning of basic facts such as multiplication (see pages 180–182), this need is demonstrated to the children through a discussion of the best ways of getting answers to multiplication questions and through a test. Naturally, there must be understanding before this second principle can be put into effect.

The third principle of drill is that the drill must be directly on the process; that is, if the job of the child is to fix $7 \times 6 = 42$, he need not spend time playing a game for which learning the rules would require some of his time. Meeting this condition almost eliminates from the drill program the use of games, devices, and problems. Since the child's attention must be on the fact or process to be learned, the setting or story element supplied by problems would be only a hindrance. However, the motivation provided by games and problems may in part compensate for the attention detracted from the process.

The fourth principle, recognition of the fact that not all arithmetical learning is important enough to require the use of drill, is relatively minor. If not observed, however, it will force the arithmetic program to put an impossible load on the learners.

If the four principles listed above are followed, the drill program becomes an effective part of arithmetic teaching. Not only are facts and processes fixed through drill, but it is through drill procedures that some of the forgotten or poorly learned facts and processes are maintained.[1]

Because drill was and still is often misused, many people associate the word with bad teaching. For this reason there are many sincere teachers and administrators who earnestly wish that arithmetic programs would eliminate drill. These people have not seen the true role of drill in arithmetic. To eliminate practice on facts or processes that the children already understand and will use over and over is to deprive them of a great time- and energy-saving procedure. After all, numbers make their contribution through elimination of long and cumbersome thought-processes. Fixation through drill is one way to eliminate cumbersome procedures. Of course, the learning of a fact or process to the point where responses become automatic can be an economical step only if the fact or process so learned is used very frequently.

While it is difficult to state with any assurance why drill is so frequently abused, the main reasons are believed to be: (1) the failure to recognize the purposes of arithmetic, and (2) the ease with which drill procedures can be applied. To

[1] For a recent discussion of the place of drill, see B R Buckingham, in *Sixteenth Yearbook*, National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1941), chap IX, pp 196–224. For an earlier statement on the criteria for drill, see F B Knight, in *Third Yearbook*, Department of Superintendents, pp 63–64. For a comprehensive discussion of drill, see J B. Stroud, "The Role of Practice in Learning," in *Forty-first Yearbook*, National Society for the Study of Education (Bloomington, Illinois: Public School Publishing Company, 1942), pp. 353–76.

decide whether or not the statements concerning drill as given in the various writings of authorities are responsible for the wrong uses of drill, the reader should consult those writings.

2. *Can teachers who have not used the method described in this book readily adopt it?*

This question is most frequently asked by supervisors and other administrators and therefore is not of direct concern to teachers. If only a categorical answer to the question were permitted, that answer would be "No." Teachers have so long been taught to demonstrate, to test, and then explain further, that it is difficult for them to permit children to take time trying to learn for themselves. On the other hand, any teacher who is willing to do some careful study of the aims of arithmetic and the procedures outlined in this book will be able to adopt the method.[1]

3. *Does not the emphasis on long, inefficient methods like counting in addition fix these procedures and thereby handicap children?*

The answer is in most cases an emphatic "No." With a few people, these longer procedures are fixed and become the only means of figuring. Instead of being a handicap for such individuals, however, the method is a definite asset. There are many people who avoid figuring entirely because they have never fully understood or mastered the processes the school has tried to teach. On the other hand, these same people could have used and understood a simpler (though longer) method and with it would be able to solve the number situations encountered in life. In other words, it is considered better for these people to use an inefficient method than to be unable to cope with number situations at all. Then, too, there is the possibility

[1] J. W. A Young, *The Teaching of Mathematics in the Elementary and the Secondary School* (New York Longmans, Green and Company, 1907), pp. 75–79. (A discussion of the difficulties teachers experience with the heuristic method.)

that they might have learned a better method had they been permitted to employ for a while a longer method which they understood.

4. *Does not the method take far more time than is available for arithmetic?*

If judgment is made on the basis of data obtained from teaching only one phase of arithmetic, the answer is definitely "Yes." However, since learning by this method is more thorough and the things learned are understood better, it is reasonable to assume that not nearly so much time is needed for maintenance. Furthermore, as children continue to learn in that way, less time is required and the advanced phases of arithmetic do not involve many new things. In weighing the amount of time given to arithmetic when a method such as that advocated in this book is used, the reader should consider additional values from the study of arithmetic other than mere subject mastery.

5. *Will the method described harden into a routine, become stereotyped and formal routine like other methods as soon as teachers become familiar with it?*

To that question an answer is difficult to frame. If teachers stop thinking about ways to interest and challenge the pupil when they become familiar with the method, then the answer is "Yes." In such a situation, however, one can hardly say that the method is used. With every new group of pupils, teaching is always a new and different process. Great teachers are all at times actors and consequently do not use the same techniques on different audiences. The teacher or supervisor who is looking for something fixed, final, as the solution to teaching difficulties will not find the answer in the method described in this book The reader of this book will recognize that most of the procedures advocated are not new. They have been used in

the past, but have lost their effectiveness because of misuse through formalization or routine application. Every method is subject to such misuses. There is no good reason for believing that the method outlined here is not subject to the same limitations.

6. *Can any part of the method be used if the use of problems with the development of new facts and processes from the pupils' experiences is omitted?*

While the use of the problem situation, with its emphasis on development of new facts and processes from the experiences of the pupils, is a major characteristic of the procedures described, teachers need not feel that this aspect of the method must be employed in order to put into practice the other procedures described. For example, the requiring of proof, the intensive study periods with emphasis upon reason for study and understanding of that which is being studied, emphasis on various methods of study, the oral arithmetic, the historical and recreational aspects of number, as well as other procedures described in this book, can be used without putting emphasis on developmental experience in initial learning procedures.

7. *What do you consider to be the main advantages of the method over other methods advocated in other books?*

Before considering the answer to this question, the reader should recall that connotations of the term *method* are many, and therefore very difficult to grasp. As was pointed out in Chapter 2, there is so much overlap between methods that contrasts and comparisons need to be made with care. For example, in the answer to question 6 above, it was pointed out that many of the procedures advocated in this book are applicable to other methods of instruction. The answer to the question, "What are the main advantages of the method?" is then subject to

the above and other limitations. Nevertheless, advantages considered to be major by the author are listed:

(1) The emphasis upon "find out for yourself," whether it be in the development of a new fact, in the consideration of ways of study, or in other situations, gives to the method a decided advantage over all others. Through this emphasis the arithmetic classroom assumes some of the naturalness, creative drive, and problem-solving attitude characteristic of elementary science and social studies classrooms. For example, a sixth-grade science class in the study of sound raised and set about finding the answer to these and other questions: (*a*) Why are some sounds high and some low? (*b*) What makes the sound in my voice? (*c*) How are sounds made? (*d*) Why does tightening my lips make a higher note on my trumpet? (*e*) How does a cold change your voice? By means of experiments (suggested by text and teacher) with vibrating sticks, rubber bands, tuning forks, resonance tubes, and by study and discussion of reading material in various sources, the pupils arrived at reasonable (satisfying to pupils and teacher) answers to the questions. Learning in such situations is characterized by the spirit of inquiry, the method of the scholar. In contrast, most learning in arithmetic classes is a case (maybe somewhat sugar-coated by play settings and make-believe) of learning facts and processes that are explained and presented by the text or teacher. Even the subject of reading, which has far less content than arithmetic, is now taught in a manner which challenges the child, arouses his attention and interest. Instead of merely reading lines or sentences, the children discuss pictures and stories with the teacher in such a way that they themselves feel they want to read in order to find out "what the girl's name is," "why she is so happy," and the like. That same spirit of learning is very much needed in arithmetic. Since the method advocated in this book makes for learning of this type, it can be seen why an awakening of the spirit of inquiry

is considered one of the important advantages that this method has over others.

(2) The method as described gives the pupil an opportunity to make full use of what he already knows. In fact, most arithmetic consists not in the learning of new things, but in the use of already known facts and generalizations in an attempt to find a simpler method of dealing with the quantitative situation. Of course, as the child progresses through the school, he is confronted with situations that involve larger and larger numbers; therefore, simple and efficient methods of handling them must be found. In this search for better methods, however, the child should be permitted, even encouraged, to start with what he already knows Thus the few simple generalizations learned in counting and in the concept-building program become the foundation for the advanced work. For example, lesson 8 in Chapter 4 (page 105) illustrates the basic idea of measurement, namely, that measures are simple things, near at hand, used to reproduce or to give an idea of the size of some object. This same idea is used again and again in the thinking of the child as he attacks measurement problems in the upper grades. The use of the simple, known facts is also well illustrated in learning to count, to add tens, and in learning to work percentage problems.

(3) The discussion on the use that the method makes of what the pupil already knows leads directly to another advantage of the method: the emphasis that it places on the use of relations through the teaching of the number system. Beginning with the teaching of counting, the program constantly emphasizes the relations that can be used if the system is followed. The child is given an opportunity to see the relation between tens and ones, in that twenty is two tens and that a collection like a pair of hands or a tens block can be used to represent a ten. In that way a ten assumes some of the properties of a one, and the child has some basis for accepting the statement that tens are

handled just as ones are. The teaching of approximation and the related proeess of reducing relations to simple ratios is another example of how knowledge of the number system can be used to make numerical situations easier to grasp.

(4) The method of study, through its emphasis on the idea that problems present all the facts so that the only task of the learner is to find the most effieient way of dealing with these faets, fosters eonfidenec and independenee on the part of the pupil. If everything is presented in the problem, the ehild who ean read or hear can draw diagrams or make pietures with objects, as the savage did, or present with numbers the situation deseribed with words in the problem. Since the procedures listed represent various levels of ability, it is almost certain that every normal child can manipulate one of them, or one closely related. Through such work the child gains eonfidence in his ability.

(5) The method of providing learning exercises which illustrate the nature and purpose of processes gives the child a logieal reason for spending his time in the study of arithmetic. For example, the multiplieation example on page 178 presents the reason for making tens and ones and the explanation of how tens and ones are made out of six groups of four each. The measurement exereises in Chapter 8 (pages 236 ff.) not only show the nature of measurement of area, but also make elear why standard measures are needed.

(6) Throughout the learning procedures advocated, the child's attention is constantly directed toward learning and beeoming aware of effieient methods of study. The emphasis direeted toward gaining an understanding of faets and proeesses and toward beeoming familiar with the purpose of such faets and proeesses makes this teaching of methods of study more logical. The learning exercises used in the method make elear to the ehild why it is important to him to learn. The diseussion of methods of learning and the frequent examination by the

teacher of methods used assure the child at least the opportunity of becoming acquainted with methods of study and certainly call to the pupil's attention this important part of school work.

(7) The method is especially commendable because of the emphasis that it places on understanding. Not only are known or familiar problem situations used to introduce new aspects of arithmetic, but the child often searches for more than one way of solving the problem. Recognizing that a problem presents all conditions and that the solution cannot therefore introduce anything new, but can only simplify, is in itself an important step toward understanding. Since in most cases the child is searching for a simpler way of presenting a fact or doing a process, the child can "see sense in what he does." The child's understanding is frequently checked by asking him to prove that his solution is correct. Through this procedure the learning program is constantly directed toward this most important step in teaching.

(8) The method recognizes and makes use of drill in learning arithmetic. In order that the major prerequisites of drill are met, heavy emphasis is first put on understanding, and then exercises are provided which will give the child an opportunity to see why certain phases of arithmetic require intensive practice or drill. In addition, through the consideration of methods of study, the child has a chance to learn the best methods of intensive study.

(9) The method presented in this book has the advantage of being more lifelike than most methods; that is, the arithmetic presented and the methods of presenting it are in agreement with the arithmetic of life as practiced by those who have learned to take advantage of the economy that the number system offers. For example, much of the arithmetic of life is done without pencil or paper. This oral or unwritten arithmetic is emphasized in the program advocated. A similar statement could be made for teaching approximation, the making of judg-

ments, the function of standards for reference, and the reduction to simple ratios of numerical quantities to be compared. Furthermore, the procedure which introduces all new facts and processes through the use of a familiar problem situation closely approaches the uses of arithmetic in life.

Because the demarcation line between what is exclusively a part of the method described in this book and methods advocated by others is difficult to establish, no further advantages will be listed. In fact, it should be noted that several of the nine points listed are claimed by others as advantages of their methods of teaching. For additional material related to this topic the reader should consult the section in Chapter 2 on "Analysis of the Sample Learning Procedures" (page 40). Irrespective of whether or not other plans do the same, the procedures advocated in this book attempt to put emphasis on meaning and understanding and at the same time make for the mastery of useful arithmetical skills in an efficient manner. By this emphasis it is hoped that children's study of arithmetic will result in their acquisition of some of the benefits of mathematics, especially the elimination of unnecessary thought, and that in their study of arithmetic children will be able to participate in learning experiences that are educationally second to none.

STUDY QUESTIONS

1. If arithmetic is taught in a meaningful way and children actually acquire understanding, is drill essential? *(1)* Yes. *(2)* No.

2. What is the chief limitation of the method of instruction which attempts to let the child develop or work out facts and procedures? *(1)* There will be too much variation in the facts and procedures developed. *(2)* There is no good way of making assignments. *(3)* The teacher will usually have to supply the fact anyway. *(4)* N.

3. How can individual differences be cared for if all members

of the class work on the same assignment; for example, all members of a third-grade class working on the same addition problems? (*1*) By requiring the fast-learning pupils to work more problems than the slow-learning pupils. (*2*) By permitting the pupils to use various methods of solving. (*3*) By allotting more teacher-time to the slow-learning pupils. (*4*) N.

4. Why is it important to try to have the oral exercise deal with content that is different from that used in the other part of the class period? (*1*) To avoid monotony. (*2*) To make introduction and recall of varied material easy. (*3*) In order not to interfere with the pupil's interest in his written work. (*4*) N.

5. Is it considered good teaching procedure to use practice or drill exercises before all members of a class understand the process to be mastered? (*1*) Yes. (*2*) No.

6. Why are problems involving a fact not as good for practice exercises as are examples? (*1*) Because the computation required is too difficult. (*2*) Because the fact is put into an uncommon setting. (*3*) Because it is too difficult to prove. (*4*) N.

7. The use of extra time required for teaching arithmetic by the method advocated in this course is partly justified by the claim that learning by this method is more permanent. What other good argument might be used to justify use of this extra time? (*1*) More facts and processes are taught by the method. (*2*) In the long run less time is used because not so much drill is needed. (*3*) The method provides valuable educational experiences other than arithmetic. (*4*) N.

Index